Logic and Languages of Higher-Dimensional Automata

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Abstract. In this paper we study finite higher-dimensional automata (HDAs) from the logical point of view. Languages of HDAs are sets of finite bounded-width interval pomsets with interfaces ($\mathsf{iiPoms}_{\leq k}$) closed under order extension. We prove that languages of HDAs are MSO-definable. For the converse, we show that the order extensions of MSO-definable sets of $\mathsf{iiPoms}_{\leq k}$ are languages of HDAs. Furthermore, both constructions are effective. As a consequence, unlike the case of all pomsets, the order extension of any MSO-definable set of $\mathsf{iiPoms}_{\leq k}$ is MSO-definable.

1 Introduction

Connections between logic and automata play a key role in several areas of theoretical computer science – logic being used to specify the behaviours of automata models in formal verification, and automata being used to prove the decidability of various logics. The first and most well-known result of this kind is the equivalence in expressive power of finite automata and monadic second-order logic (MSO) over finite words, proved independently by Büchi [5], Elgot [10] and Trakhtenbrot [37] in the 60's. This was soon extended to infinite words [6] as well as finite and infinite trees [8, 32, 34].

Finite automata over words are a simple model of sequential systems with a finite memory, each word accepted by the automaton corresponding to an execution of the system. For concurrent systems, executions may be represented as *pomsets* (partially ordered multisets or, equivalently, labelled partially ordered sets). Several classes of pomsets and matching automata models have been defined in the literature, corresponding to different communication models or different views of concurrency. In that setting, logical characterisations of classes of automata in the spirit of the Büchi-Elgot-Trakhtenbrot theorem have been obtained for several cases, such as asynchronous automata and Mazurkiewicz traces [35, 40], branching automata and series-parallel pomsets [3, 29], step transition systems and local trace languages [21, 30], or communicating finite-state machines and message sequence charts [23].

Higher-dimensional automata (HDAs) [31, 38] are another automaton-based model of concurrent systems that matches more closely an interval-based view of events. Initially studied from a geometrical or categorical point of view, the

language theory of HDAs has become another focus for research in the past few years [13]. The language of an HDA is defined as a set of *interval pomsets with interfaces (interval ipomsets)* [15]. The idea is that each event in the execution of an HDA corresponds to an interval of time where some process is active.

Examples with three activity intervals labelled a, b, and c are shown in the top of Fig. 1 below. These events are then partially ordered as follows: two events are ordered if the first one ends before the second one starts, and they are concurrent if they overlap. This gives rise to a pomset as shown in the middle of Fig. 1. We allow some events to be started before the beginning (this is the case for the a-labelled events in Fig. 1), and some events might never be terminated. Such events are called interfaces, or respectively sources and targets.

In addition, if we shorten some intervals in one possible behaviour of the HDA, we obtain another valid behaviour for the HDA. In terms of pomsets, this means that the language of an HDA is closed under *subsumption* (called *order extension* in [21]). In addition, it also has bounded width, meaning that each set of pairwise concurrent events has size at most k for some k.

The interest of HDAs as a model for concurrency stems from their convenient automata-theoretic and geometric properties. They provide a natural extension of standard automata to higher dimensions and lend themselves to automata-theoretic reasoning. Several theorems of classical automata theory have already been extended to HDAs, including a Kleene theorem [14] and a Myhill-Nerode theorem [17], and [1, 12] provide an extension to higher-dimensional *timed* automata. On the other hand, the precubical sets on which HDAs are based have been studied in geometry and algebraic topology for a long time [4, 25, 33] and have led to interesting results for example about state-space reduction [16, 18, 20, 41] or about behavioural equivalences [9, 11, 27, 28], see [19] for a recent overview.

The automata-theoretic closure properties of HDAs were studied in [2]. In particular, their languages are not closed under complement, but they are closed under *bounded width complement*: the subsumption closure of the complement of the language restricted to interval ipomsets of bounded width.

In this paper, we explore the relationship between HDAs and MSO. We prove that a set of interval ipomsets is regular if and only if it is simultaneously MSO-definable, of bounded width, and downward-closed for subsumption. The latter two assumptions are necessary as it is possible to define in MSO sets with unbounded width or sets that are not downward-closed.

The HDA-to-MSO direction is proved similarly to the original Büchi-Elgot-Trakhtenbrot theorem. We use one second-order variable for each *upstep* (starting events) or *downstep* (terminating events) of the HDA. The main difference with words is that each upstep or downstep involves several events. We rely on the existence of a canonical *sparse step decomposition* for any interval ipomset. We prove that this decomposition can be effectively "defined" in MSO.

On the other hand, the usual approach for the MSO-to-automata direction, which works by induction and relies on the closure properties of regular languages, does not work for HDAs, as they are not closed under complement.

One could try to use the bounded-width complement instead, but the downward closures present some difficulties. Instead, we rely on a known connection [2] between regular languages of interval ipomsets and regular languages of *step decompositions*. A step decomposition of an ipomset P is a sequence of discrete ipomsets (that is, pomsets where all events are concurrent) such that their composition is equal to P. We prove that for every MSO-definable language L of width at most k, the language of all step decompositions of ipomsets in L, viewed as words over a finite alphabet of discrete ipomsets, is regular. To do so, we give an effective translation from MSO formulas over ipomsets to MSO formulas over words with this new alphabet. It was shown in [14] that the downward closure of L is then effectively regular.

The paper is organised as follows. Interval pomsets with interfaces and step decompositions are defined in Section 2, and higher-dimensional automata in Section 3. In Section 4, we introduce monadic second-order logic and state our main result. Section 5 gives the proof for the MSO-to-HDA direction, and Section 6 for the HDA-to-MSO one.

2 Pomsets with Interfaces

We fix a finite alphabet Σ throughout this paper. A *pomset with interfaces*, or *ipomset*, is a structure $(P, <, \dashrightarrow, S, T, \lambda)$ comprising a finite set P, a (strict) partial order[3] $< \subseteq P \times P$ called the *precedence order*, an irreflexive and asymmetric relation $\dashrightarrow \subseteq P \times P$ called the *event order*, subsets $S, T \subseteq P$ called *source* and *target* sets, and a *labelling* $\lambda\colon P \to \Sigma$. We require the following properties:

- for all $e \neq e' \in P$, exactly one of $e < e'$, $e' < e$, $e \dashrightarrow e'$, or $e' \dashrightarrow e$ holds;
- for all $e_1 \in S$, $e_2 \in P$, and $e_3 \in T$, $e_2 \not< e_1$ and $e_3 \not< e_2$.

That is, all points in P are related by precisely one of the orders, sources are $<$-minimal, and targets are $<$-maximal. We may add subscripts "$_P$" to the elements above if necessary. The source and target *interfaces* of P are the conclists $(S, \dashrightarrow_{\restriction S \times S}, \lambda_{\restriction S})$ and $(T, \dashrightarrow_{\restriction T \times T}, \lambda_{\restriction T})$, where "$\restriction$" denotes restriction.

Ipomsets are a generalisation of standard pomsets (see for example [24]) obtained by adding interfaces and event order. Both are needed in order to properly connect them with HDAs. In particular, event order is necessary in order to define gluing composition, see below. Further, a central tool in the language theory of HDAs (which we, however, do not need here) are so-called track objects, which provide an embedding of ipomsets into HDAs which essentially needs the event order; see [13] for details. In [13] and other works, a transitively closed event order is used instead of the relation we use here; we find it more convenient to use the non-transitive version which otherwise is equivalent.

An ipomset P is a *word* (with interfaces) if $<$ is total and *discrete* if $< = \emptyset$. P is a *pomset* if $S = T = \emptyset$, a *conclist* (short for "concurrency list") if it is a discrete pomset, a *starter* if it is discrete and $T = P$, a *terminator* if it is discrete and $S = P$, and an *identity* if it is both a starter and a terminator.

[3] A strict partial order is a relation which is irreflexive, asymmetric and transitive. We will omit the qualifier "strict".

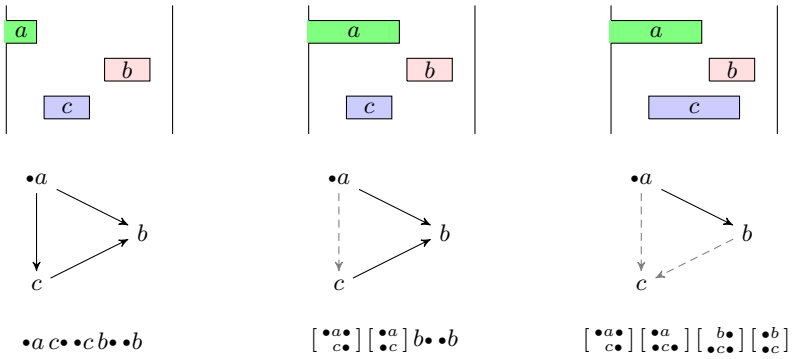


Fig. 1: Activity intervals of events (top), corresponding ipomsets (middle), and notation as a word over starters and terminators (bottom), *cf.* Ex. 1. Full arrows indicate precedence order; dashed arrows indicate event order; bullets indicate interfaces.

Figure 1 shows some simple examples. Source and target events are marked by "•" at the left or right side. Precedence $<$ and event order $\dashrightarrow$ are intended to order sequential and concurrent events, respectively. Another representation that we will use is as words of starters and terminators. In this representation, the ipomset is decomposed into a sequence of starters and terminators representing which events are in parallel and in which order. In the starter/terminator representation, event order is omitted and always assumed to go downward.

Ipomsets P and Q are *isomorphic* if there exists a bijection $f\colon P \to Q$ for which

1. $f(S_P) = S_Q$, $f(T_P) = T_Q$, $\lambda_Q \circ f = \lambda_P$, and
2. $f(e_1) <_Q f(e_2) \iff e_1 <_P e_2$ and $e_1 \dashrightarrow_P e_2 \iff f(e_1) \dashrightarrow_Q f(e_2)$.

That is, f respects interfaces and labels and the two relations. Because of the requirement that all elements are related by $<$ or $\dashrightarrow$, there is at most one isomorphism between any two ipomsets [13]. That means that we may switch between ipomsets and their isomorphism classes, and we will do so often in the sequel.

An ipomset P is *interval* if $<_P$ is an interval order [22]; that is, if it admits an interval representation given by functions $f, g\colon (P, <_P) \to (\mathbb{R}, <_{\mathbb{R}})$ such that $f(e) \leq_{\mathbb{R}} g(e)$ for all $e \in P$ and $e_1 <_P e_2$ iff $g(e_1) <_{\mathbb{R}} f(e_2)$ for all $e_1, e_2 \in P$. Given that our ipomsets represent activity intervals of events, any of the ipomsets we will encounter will be interval, and we omit the qualification "interval". We emphasise that this is *not* a restriction, but rather induced by the semantics, [39]. The *width* $\mathsf{wid}(P)$ of an ipomset P is the cardinality of a maximal $<$-antichain.

We let iiPoms denote the set of (interval) ipomsets and $\mathsf{iiPoms}_{\leq k} = \{P \in \mathsf{iiPoms} \mid \mathsf{wid}(P) \leq k\}$. We write $\mathsf{St}, \mathsf{Te}, \mathsf{Id} \subseteq \mathsf{iiPoms}$ for the sets of starters, terminators, and identities and let $\Omega = \mathsf{St} \cup \mathsf{Te}$. Further, for $S \in \{\mathsf{St}, \mathsf{Te}, \mathsf{Id}, \Omega\}$, $S_{\leq k} = S \cap \mathsf{iiPoms}_{\leq k}$. Note that $\mathsf{Id} = \mathsf{St} \cap \mathsf{Te}$ and $\mathsf{Id}_{\leq k} = \mathsf{St}_{\leq k} \cap \mathsf{Te}_{\leq k}$.

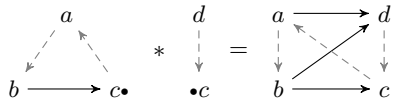


Fig. 2: Gluing composition of ipomsets.

We introduce special notation for starters and terminators and write ${}_A{\uparrow}U = (U, \emptyset, \dashrightarrow, U \setminus A, U, \lambda)$ and $U{\downarrow}_B = (U, \emptyset, \dashrightarrow, U, U \setminus B, \lambda)$ (with $\dashrightarrow$ and λ induced from U). The intuition is that the starter ${}_A{\uparrow}U$ does nothing but start the events in $A = U \setminus S_U$ and the terminator $U{\downarrow}_B$ terminates the events in $B = U \setminus T_U$.

Ipomsets may be *refined* by shortening activity intervals, potentially removing concurrency and expanding precedence. The inverse to refinement is called *subsumption*. Formally, for ipomsets P and Q we say that P refines Q, or that Q subsumes P, and write $P \sqsubseteq Q$ if there is a bijection $f\colon P \to Q$ for which

(1) $f(S_P) = S_Q$, $f(T_P) = T_Q$, and $\lambda_Q \circ f = \lambda_P$,
(2) $f(e_1) <_Q f(e_2) \implies e_1 <_P e_2$, and $e_1 \dashrightarrow_P e_2 \implies f(e_1) \dashrightarrow_Q f(e_2)$.

This definition adapts the one of [24] to event orders and interfaces. Intuitively, P has more order and less concurrency than Q. Note that isomorphisms are precisely those subsumptions whose inverses are also subsumptions.

For a subset $A \subseteq \mathsf{iiPoms}$ we let

$$A{\downarrow} = \{P \in \mathsf{iiPoms} \mid \exists Q \in A : P \sqsubseteq Q\}$$

denote its closure under subsumptions. A *language* is a subset $L \subseteq \mathsf{iiPoms}$ for which $L{\downarrow} = L$.

Example 1. In Fig. 1 there is a sequence of subsumptions from left to right. An event e_1 is smaller than e_2 in the precedence order if e_1 is terminated before e_2 is started; e_1 is smaller than e_2 in the event order if they are concurrent and e_1 is above e_2 in the respective conclist.

The *gluing* $P * Q$ of ipomsets P and Q is defined if $T_P = S_Q = P \cap Q$ *as conclists* (*i.e.*, $\dashrightarrow_{P{\restriction}T_P \times T_P} = \dashrightarrow_{Q{\restriction}S_Q \times S_Q}$ and $\lambda_{P{\restriction}T_P} = \lambda_{Q{\restriction}S_Q}$), and then $P * Q = (P \cup Q, <, \dashrightarrow, S_P, T_Q, \lambda)$, where $< = <_P \cup <_Q \cup (P \setminus T_P) \times (Q \setminus S_Q)$, $\dashrightarrow = (\dashrightarrow_P \cup \dashrightarrow_Q)^+$, and $\lambda = \lambda_P \cup \lambda_Q$. (Here $^+$ denotes transitive closure.) Gluing is associative, and ipomsets in Id are identities for $*$. Figure 2 shows an example.

Any ipomset P can be decomposed as a gluing of starters and terminators $P = P_1 * \cdots * P_n$ [15, 26]. Such a presentation we call a *step decomposition*. If starters and terminators are alternating, the step decomposition is called *sparse*.

Lemma 2 ([17]). *Every ipomset P has a unique sparse step decomposition.*

We will also use the following notion, introduced in [2]. A word $P_1 \dots P_n \in \Omega^*$ is *coherent* if the gluing $P_1 * \cdots * P_n$ is defined. We denote by $\mathsf{Coh} \subseteq \Omega^*$ the set of coherent words and $\mathsf{Coh}_{\leq k} = \mathsf{Coh} \cap \Omega^*_{\leq k}$.

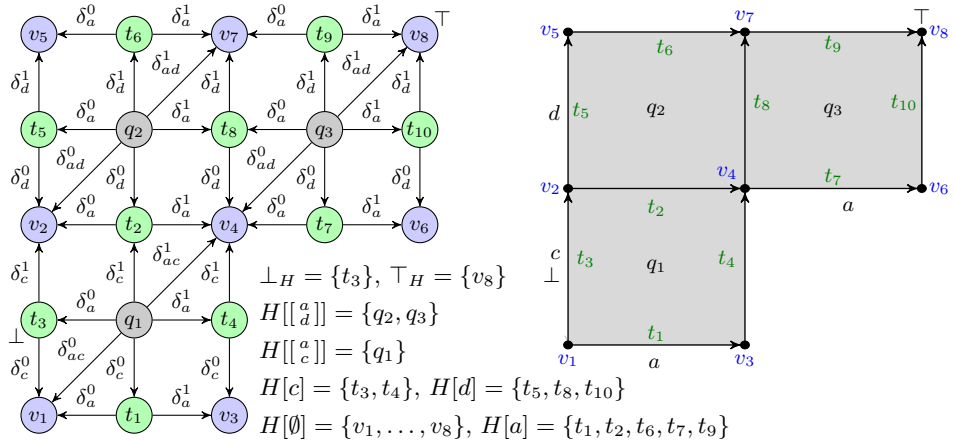


Fig. 3: A two-dimensional HDA $\mathcal{H}$ on $\Sigma = \{a, c, d\}$, see Ex. 3.

3 Higher-dimensional automata

Let $\square$ denote the set of conclists. A *precubical set*

$$\mathcal{H} = (H, \mathsf{ev}, \{\delta^0_{A,U}, \delta^1_{A,U} \mid U \in \square, A \subseteq U\})$$

consists of a set of *cells* H together with a function $\mathsf{ev}\colon H \to \square$ which to every cell assigns a conclist of concurrent events which are active in it. We write $H[U] = \{q \in H \mid \mathsf{ev}(q) = U\}$ for the cells of type U. For every $U \in \square$ and subset $A \subseteq U$ there are *face maps* $\delta^0_{A,U}, \delta^1_{A,U}\colon H[U] \to H[U \setminus A]$ (we omit the subscript U from now) which are required to satisfy $\delta^\nu_A \delta^\mu_B = \delta^\mu_B \delta^\nu_A$ for $A \cap B = \emptyset$ and $\nu, \mu \in \{0, 1\}$. The *upper* face maps δ^1_A terminate events in A and the *lower* face maps δ^0_A transform a cell q into one in which the events in A have not yet started. A *higher-dimensional automaton* (*HDA*) $\mathcal{H} = (\mathcal{H}, \bot_H, \top_H)$ is a finite precubical set together with subsets $\bot_H, \top_H \subseteq H$ of *start* and *accept* cells. The *dimension* of an HDA $\mathcal{H}$ is $\dim(\mathcal{H}) = \max\{|\mathsf{ev}(q)| \mid q \in H\} \in \mathbb{N}$.

A standard automaton is the same as a one-dimensional HDA $\mathcal{H}$ with the property that for all $q \in \bot_H \cup \top_H$, $\mathsf{ev}(q) = \emptyset$: cells in $H[\emptyset]$ are states, cells in $H[\{a\}]$ for $a \in \Sigma$ are a-labelled transitions, and face maps $\delta^0_{\{a\}}$ and $\delta^1_{\{a\}}$ attach source and target states to transitions. In contrast to ordinary automata we allow start and accept *transitions* instead of merely states, so languages of one-dimensional HDAs may contain words with interfaces.

Example 3. Figure 3 shows a two-dimensional HDA as a combinatorial object (left) and in a geometric realisation (right). It consists of 21 cells: states $H_0 = \{v_1, \dots, v_8\}$ in which no event is active ($\mathsf{ev}(v_i) = \emptyset$), transitions $H_1 = \{t_1, \dots, t_{10}\}$ in which one event is active (*e.g.*, $\mathsf{ev}(t_3) = \mathsf{ev}(t_4) = c$), squares $H_2 = \{q_1, q_2, q_3\}$ where $\mathsf{ev}(q_1) = [\begin{smallmatrix}a\\c\end{smallmatrix}]$ and $\mathsf{ev}(q_2) = \mathsf{ev}(q_3) = [\begin{smallmatrix}a\\d\end{smallmatrix}]$. The arrows between cells in the left representation correspond to the face maps connecting them. For example,

the upper face map δ^1_{ac} maps q_1 to v_4 because the latter is the cell in which the active events a and c of q_1 have been terminated. On the right, face maps are used to glue cells, so that for example $\delta^1_{ac}(q_1)$ is glued to the top right of q_1. In this and other geometric realisations, when we have two concurrent events a and c with $a \dashrightarrow c$, we will draw a horizontally and c vertically.

Computations of HDAs are *paths*, *i.e.*, sequences $\alpha = (q_0, \varphi_1, q_1, \dots, q_{n-1}, \varphi_n, q_n)$ consisting of cells $q_i \in H$ and symbols φ_i which indicate face map types: for every $i \in \{1, \dots, n\}$, $(q_{i-1}, \varphi_i, q_i)$ is either

- $(\delta^0_A(q_i), \nearrow^A, q_i)$ for $A \subseteq \mathsf{ev}(q_i)$ (an *upstep*)
- or $(q_{i-1}, \searrow_A, \delta^1_A(q_{i-1}))$ for $A \subseteq \mathsf{ev}(q_{i-1})$ (a *downstep*).

Downsteps terminate events, following upper face maps, whereas upsteps start events by following inverses of lower face maps. We denote by $\mathsf{ups}(\mathcal{H})$ and $\mathsf{downs}(\mathcal{H})$ the finite set of upsteps and downsteps of $\mathcal{H}$.

The *source* and *target* of α as above are $\mathsf{src}(\alpha) = q_0$ and $\mathsf{tgt}(\alpha) = q_n$. A path α is *accepting* if $\mathsf{src}(\alpha) \in \bot_H$ and $\mathsf{tgt}(\alpha) \in \top_H$. Paths α and β may be concatenated if $\mathsf{tgt}(\alpha) = \mathsf{src}(\beta)$; their concatenation is written $\alpha * \beta$.

Path equivalence is the congruence $\simeq$ generated by $(q \nearrow^A r \nearrow^B p) \simeq (q \nearrow^{A \cup B} p)$, $(p \searrow_A r \searrow_B q) \simeq (p \searrow_{A \cup B} q)$, and $\gamma\alpha\delta \simeq \gamma\beta\delta$ whenever $\alpha \simeq \beta$. This relation allows to assemble subsequent upsteps or downsteps into one bigger step.

The *event ipomset* $\mathsf{ev}(\alpha)$ of a path α is defined recursively as follows:

- if $\alpha = (q)$, then $\mathsf{ev}(\alpha) = \mathsf{id}_{\mathsf{ev}(q)}$;
- if $\alpha = (q \nearrow^A p)$, then $\mathsf{ev}(\alpha) = {}_A\uparrow\mathsf{ev}(p)$;
- if $\alpha = (p \searrow_B q)$, then $\mathsf{ev}(\alpha) = \mathsf{ev}(p)\downarrow_B$;
- if $\alpha = \alpha_1 * \cdots * \alpha_n$ is a concatenation, then $\mathsf{ev}(\alpha) = \mathsf{ev}(\alpha_1) * \cdots * \mathsf{ev}(\alpha_n)$.

Note that upsteps in α correspond to starters in $\mathsf{ev}(\alpha)$ and downsteps correspond to terminators. Path equivalence $\alpha \simeq \beta$ implies $\mathsf{ev}(\alpha) = \mathsf{ev}(\beta)$ [14].

Example 4. The HDA $\mathcal{H}$ of Ex. 3 (Fig. 3) admits several accepting paths, for example $t_3 \nearrow^a q_1 \searrow_c t_2 \nearrow^d q_2 \searrow_a t_8 \nearrow^a q_3 \searrow_{ad} v_8$. Its event ipomset is

$${}_a\uparrow\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] * \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]\downarrow_c * {}_d\uparrow\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right] * \left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]\downarrow_a * {}_a\uparrow\left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right] * \left[\begin{smallmatrix} a \\ d \end{smallmatrix}\right]\downarrow_{ad} = \left[\begin{array}{ccc} a & \longrightarrow & a \\ \downarrow & \nearrow & \downarrow \\ \bullet c & \longrightarrow & d \end{array}\right]$$

which is a sparse step decomposition. This path is equivalent to $t_3 \nearrow^a q_1 \searrow_c t_2 \nearrow^d q_2 \searrow_a t_8 \nearrow^a q_3 \searrow_a t_{10} \searrow_d v_8$ which induces the coherent word w_1 of Fig. 4 below.

The *language* of an HDA $\mathcal{H}$ is $L(\mathcal{H}) = \{\mathsf{ev}(\alpha) \mid \alpha \text{ accepting path in } \mathcal{H}\}$. A language is *regular* if it is the language of a finite HDA. Languages of HDAs are closed under subsumption, that is, if L is regular, then $L\downarrow = L$ [13, 14].

A language is *rational* if it is constructed from $\emptyset$, $\{\mathsf{id}_\emptyset\}$ and discrete ipomsets using $\cup$, $*$ and $^+$ (Kleene plus) [14]. These operations have to take subsumption closure into account; in particular,

$$L_1 * L_2 = \{P * Q \mid P \in L_1, Q \in L_2\}\downarrow.$$

Theorem 5 ([14]). *A language is regular if and only if it is rational.*

The *width* of a language L is $\mathsf{wid}(L) = \sup\{\mathsf{wid}(P) \mid P \in L\}$. For $k \geq 0$ and $L \subseteq \mathsf{iiPoms}$, denote $L_{\leq k} = \{P \in L \mid \mathsf{wid}(P) \leq k\}$.

Lemma 6 ([14]). *Any regular language has finite width.*

It immediately follows that the universal language iiPoms is *not* rational.

4 MSO

Monadic second-order (MSO) logic is an extension of first-order logic allowing to quantify existentially and universally over elements as well as subsets of the domain of the structure. It uses second-order variables $X, Y, \dots$ interpreted as subsets of the domain in addition to the first-order variables $x, y, \dots$ interpreted as elements of the domain of the structure, and a new binary predicate $x \in X$ interpreted commonly. We refer the reader to [36] for more details about MSO.

We interpret MSO over iiPoms. Thus we consider the signature $\mathcal{S} = \{<, \dashrightarrow, (a)_{a\in\Sigma}, s, t\}$ where $<$ and $\dashrightarrow$ are binary relation symbols and the a's, s and t are unary predicates (over first-order variables). We associate to every ipomset $(P, <, \dashrightarrow, S, T, \lambda)$ the relational structure $S = (P; <; \dashrightarrow; (a)_{a\in\Sigma}; \mathsf{s}; \mathsf{t})$ where $<$ and $\dashrightarrow$ are interpreted as the orderings $<$ and $\dashrightarrow$ over P, and $a(x)$, $\mathsf{s}(x)$ and $\mathsf{t}(x)$ hold respectively if and only if $\lambda(x) = a$, $x \in S$ and $x \in T$. We say that a relation $R \subseteq P^n \times (2^P)^m$ is *MSO-definable* in S if and only if there exists an MSO-formula $\psi(x_1, \dots, x_n, X_1, \dots, X_m)$ over $\mathcal{S}$ which is satisfied if and only if the interpretation of the free variables $(x_1, \dots, x_n, X_1, \dots, X_m)$ in S is a tuple of R. The well-formed MSO formulas are built using the following grammar:

$$\psi ::= a(x) \mid \mathsf{s}(x) \mid \mathsf{t}(x) \mid x < y \mid x \dashrightarrow y \mid x \in X$$
$$\exists x.\, \psi \mid \forall x.\, \psi \mid \exists X.\, \psi \mid \forall X.\, \psi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \neg\psi$$

In order to shorten formulas we use several notations and shortcuts such as $\psi_1 \implies \psi_2$. We define $x \prec y := x < y \wedge \neg(\exists z. x < z < y)$.

Let $\psi(x_1, \dots, x_n, X_1, \dots, X_m)$ be an MSO formula whose free variables are $x_1, \dots, x_n, X_1, \dots, X_m$ and let $P \in \mathsf{iiPoms}$. The pair of functions $\nu = (\nu_1, \nu_2)$ where $\nu_1 \colon \{x_1, \dots, x_n\} \to P$ and $\nu_2 \colon \{X_1, \dots, X_m\} \to 2^P$ is called a *valuation* or an *interpretation*. We write $P \models_\nu \psi$, or, by a slight abuse of notation, $P \models \psi\big(\nu(x_1), \dots, \nu(x_n), \nu(X_1), \dots, \nu(X_m)\big)$, if ψ holds when x_i and X_j are interpreted as $\nu(x_i)$ and $\nu(X_j)$. A *sentence* is a formula without free variables. In this case no valuation is needed. Given an MSO sentence ψ, we define $L(\psi) = \{P \in \mathsf{iiPoms} \mid P \models \psi\}$. Note that this may not be closed under subsumption, hence not a language in our sense. A set $L \subseteq \mathsf{iiPoms}$ is MSO-definable if and only if there exists an MSO sentence ψ over $\mathcal{S}$ such that $L = L(\psi)$.

Example 7. Let $\varphi = \exists x\, \exists y.\, a(x) \wedge b(y) \wedge \neg(x < y) \wedge \neg(y < x)$. That is, there are at least two concurrent events, one labelled a and the other b. $L(\varphi)$ is not width-bounded, as φ is satisfied, among others, by any conclist which contains

at least one a and one b, nor closed under subsumption, given that $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \models \varphi$ but $ab, ba \not\models \varphi$. Note, however, that $L(\varphi)_{\leq k}{\downarrow}$ is a regular language for any k.

We will also use MSO over words of $\Omega^*_{\leq k}$. (Note that, up to isomorphisms, $\Omega_{\leq k}$ is a finite set.) The definitions above can be easily adapted to this case by considering a word of $\Omega^*_{\leq k}$ as a structure of the form $(W, <, \lambda\colon W \to \Omega_{\leq k})$: a totally ordered set W labelled by $\Omega_{\leq k}$, and the signature $\{<, (D)_{D \in \Omega_{\leq k}}\}$: the atomic predicates are $D(x)$ for $D \in \Omega_{\leq k}$, $x < y$ and $x \in X$, with first-order variables ranging over positions in the word and second-order variables over sets of positions. We denote by MSO^k_Ω the set of MSO formulas over $\Omega^*_{\leq k}$. For example the following MSO^2_Ω formula where $P_i \in \Omega_{\leq 2}$ stands for the ith discrete ipomset of w_1 in Fig. 4 is satisfied only by w_1.

$$\varphi' := \exists y_1, \dots, y_7. \bigwedge_{1 \leq i \leq 7} P_i(y_i) \wedge y_1 \prec \cdots \prec y_7 \wedge \forall y. \bigvee_{1 \leq i \leq 7} y = y_i$$

The main result of this paper is the following:

Theorem 8. *For all $L \subseteq \mathsf{iiPoms}$,*

1. *if L is MSO-definable, then $L_{\leq k}{\downarrow}$ is regular for all $k \in \mathbb{N}$.*
2. *if L is regular, then it is MSO-definable.*

Moreover, the constructions are effective in both directions.

Corollary 9. *For all $k \in \mathbb{N}$, a language $L \subseteq \mathsf{iiPoms}_{\leq k}$ is regular if and only if it is MSO-definable.*

The next two sections are devoted to the proof of Thm. 8. For the first assertion we effectively build an HDA $\mathcal{H}$ from a sentence φ and an integer k such that $L(\mathcal{H}) = L(\varphi)_{\leq k}{\downarrow}$. Since emptiness of HDAs is decidable [2], asking, given a formula φ such that $L(\varphi) = L(\varphi)_{\leq k}{\downarrow}$, if there exists $P \in \mathsf{iiPoms}$ such that $P \models \varphi$ and if $L(\mathcal{H}) \subseteq L(\varphi)$ for some HDA $\mathcal{H}$ are decidable.

Corollary 10. *For MSO sentences φ such that $L(\varphi) = L(\varphi)_{\leq k}{\downarrow}$, the satisfiability problem and the model-checking problem for HDAs are both decidable.*

Actually, looking more closely at our construction which goes through finite automata accepting step sequences, we get the same result for MSO formulas even without the assumption that $L(\varphi)$ is downward-closed (but still over $\mathsf{iiPoms}_{\leq k}$, and not iiPoms). This could also be shown alternatively by observing that $\mathsf{iiPoms}_{\leq k}$ has bounded treewidth (in fact, even bounded pathwidth), and applying Courcelle's theorem [7]. In fact our implied proof of decidability is relatively similar, using step sequences instead of path decompositions.

For the second assertion of the theorem, we show that regular languages of HDAs are MSO-definable, again using an effective construction. Thus, using both directions of Thm. 8 and the closure properties of HDAs, we also get the following:

Corollary 11. *For all $k \in \mathbb{N}$ and MSO-definable $L \subseteq \mathsf{iiPoms}_{\leq k}$, $L{\downarrow}$ is MSO-definable.*

Note that this property does *not* hold for the class of *all* pomsets [21].

5 From MSO to HDAs

Given an MSO sentence φ over iiPoms we effectively build an HDA $\mathcal{H}$ such that $L(\mathcal{H}) = L(\varphi)_{\leq k}{\downarrow}$. The first step is to define an MSO-interpretation of interval ipomsets of width at most k into words of $\Omega^+_{\leq k}$, so that:

Lemma 12. *For every MSO sentence φ over iiPoms and every k there exists $\widehat{\varphi} \in \mathrm{MSO}^k_\Omega$ such that for all $P_1 \dots P_n \in (\Omega_{\leq k} \setminus \{\mathsf{id}_\emptyset\})^+$, we have $P_1 \dots P_n \models \widehat{\varphi}$ if and only if $P = P_1 * \dots * P_n$ is well-defined and $P \models \varphi$.*

We will treat the case of the empty ipomset $\mathsf{id}_\emptyset$ separately afterwards. Prior to proving the lemma, we introduce key concepts and provide an overview. Informally speaking, we want a word $P_1 \dots P_n$ of $(\Omega_{\leq k} \setminus \{\mathsf{id}_\emptyset\})^+$ to satisfy $\widehat{\varphi}$ if and only if the gluing composition $P = P_1 * \dots * P_n$ is a model for φ. Thus $\widehat{\varphi}$ must accept only coherent words. This is MSO^k_Ω-definable by:

$$\mathtt{Coh}_k := \forall x \, \forall y. \, x \prec y \implies \bigvee_{P_1 P_2 \in \mathsf{Coh}_{\leq k} \cap \Omega^2_{\leq k}} P_1(x) \wedge P_2(y).$$

That is, discrete ipomsets of $\Omega_{\leq k}$ at consecutive positions x and y may be glued.

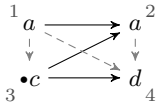
Hence, $\widehat{\varphi}$ will be the conjunction of $\mathtt{Coh}_k$ and an MSO^k_Ω formula φ' which we will build by induction on φ. Therefore, we have to consider formulas φ that contain free variables. We will construct φ' so that its free variables will be all the free first-order variables of φ and second-order variables $X_1, \dots, X_k$ for every free second-order variable X of φ. In addition, we make sure during the construction that when $P \models \varphi$ then for every $w = P_1 \dots P_n$ such that $P_1 * \dots * P_n = P$, when the free variable x is interpreted as some event e of P in φ it is interpreted in φ' as some position in w where e occurs, and X_i will contain positions of w having as ith event according to $\dashrightarrow$ the interpretation of some element of X (see Ex. 13).

More formally, let $w = P_1 \dots P_n \in \mathsf{Coh}_{\leq k}$ and $P = P_1 * \dots * P_n$. Let $E = \{1, \dots, n\} \times \{1, \dots, k\}$. Our construction is built on a partial function $evt \colon E \to P$ defined as follows: if P_ℓ consists of events $e_1 \dashrightarrow \dots \dashrightarrow e_r$, then for every $i \leq r$, $evt(\ell, i) = e_i$. We sometimes abuse notation and write $evt(P_\ell, i)$. Since $e \in P$ may occur in consecutive P_ℓ within w, one must determine when $evt(\ell, i) = evt(\ell', j)$. This can be done in MSO^k_Ω when $\ell' = \ell + 1$ as follows. For all $i, j \leq k$, let $M_{i,j} = \{P_1 P_2 \in \Omega^2_{\leq k} \mid evt(1, i) = evt(2, j)\}$. Then

$$\mathit{glue}_{i,j}(x, y) := x \prec y \wedge \bigvee_{P_1 P_2 \in M_{i,j}} P_1(x) \wedge P_2(y) \,.$$

More generally, let us define the equivalence relation $\sim$ on E generated by $(\ell, i) \sim (\ell', i')$ if and only if $\mathit{glue}_{i,i'}(\ell, \ell')$ holds. Then for all $(\ell_1, i), (\ell_2, j) \in E$, $(\ell_1, i) \sim (\ell_2, j)$ if and only if $evt(\ell_1, i) = evt(\ell_2, j)$. The relation $\sim$ is MSO^k_Ω-definable:

$$(x, i) \sim (y, j) := \forall X_1, \dots, X_k. \Big(x \in X_i \wedge \textstyle\bigwedge_{i,j \leq k} \forall x, y. \\ x \in X_i \wedge (\mathit{glue}_{i,j}(x, y) \vee \mathit{glue}_{j,i}(y, x)) \implies y \in X_j \Big) \implies y \in X_j \,.$$



$$w_1 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \begin{bmatrix} a\bullet \\ \bullet c\bullet \end{bmatrix} \begin{bmatrix} \bullet a\bullet \\ \bullet c \end{bmatrix} \begin{bmatrix} \bullet a\bullet \\ d\bullet \end{bmatrix} \begin{bmatrix} \bullet a \\ \bullet d\bullet \end{bmatrix} \begin{bmatrix} a\bullet \\ \bullet d\bullet \end{bmatrix} \begin{bmatrix} \bullet a \\ \bullet d\bullet \end{bmatrix} [\bullet d]$$

Fig. 4: Ipomset and corresponding coherent word (numbers indicate positions).

As mentioned before, we want a free first-order variable x to be interpreted in φ' as a position $P_\ell \in \Omega_{\leq k}$ of a coherent word $P_1 \dots P_n$ where the interpretation $e \in P_1 * \dots * P_n$ of x in φ occurs. That is there exists i such that $evt(l, i) = e$. Precisely, we construct a formula φ'_τ relative to a function τ which associates with every free first-order variable x of φ some $\tau(x) \in \{1, \dots, k\}$. We sometimes leave τ implicit. Our aim is to have the following *invariant property* at each step of the induction: $P \models_\nu \varphi$ if and only if $w \models_{\nu'} \varphi'_\tau$ for any valuations ν, ν' satisfying the following:

1. $evt(\nu'(x), \tau(x)) = \nu(x)$ and
2. $\bigcup_{1 \leq i \leq k} \{evt(e, i) \mid e \in \nu'(X_i)\} = \nu(X)$.

Example 13. Figure 4 displays an ipomset P and the coherent word $w_1 = P_1 \dots P_7$ such that $P_1 * \dots * P_7 = P$. Let $e_1, \dots, e_4$ be the events of P labelled respectively by the left a, the right a, c, and d and let $p_1, \dots, p_7$ the positions on w_1 from left to right. Assume that $P \models_\nu \varphi(x, X)$ for some MSO-formula φ and the valuation $\nu(x) = e_1$ and $\nu(X) = \{e_2, e_3\}$. Then, $w_1 \models_{\nu'} \varphi'_{[x \mapsto 1]}(x, X_1, X_2)$ when, for example, $\nu'(x) = p_2$, $\nu'(X_1) = \{p_6\}$ and $\nu'(X_2) = \{p_3\}$ since this valuation satisfies the invariant property. For $\sim$ we have $(p_1, 1) \sim \dots \sim (p_4, 1)$, $(p_1, 2) \sim (p_2, 2)$, $(p_3, 2) \sim \dots \sim (p_6, 2) \sim (p_7, 1)$ and $(p_5, 1) \sim (p_6, 1)$. In particular $(p_1, 1) \not\sim (p_5, 1)$ since neither $glue_{1,1}(p_4, p_5)$ nor $glue_{2,1}(p_4, p_5)$ hold.

We are now ready to prove Lem. 12.

Proof (of Lem. 12). First, we let $\widehat{\varphi} := \mathtt{Coh}_k \wedge \varphi'$ and we build φ' by induction on φ as follows. When φ is $\psi_1 \vee \psi_2$ or $\neg\psi$, then we let φ' be $\psi'_1 \vee \psi'_2$ or $\neg\psi'$, respectively. For $\varphi = \exists X\, \psi$ we let $\varphi' := \exists X_1, \dots, X_k.\psi'$. The function τ emerges in the case $\varphi = \exists x\, \psi$, where we let $\varphi'_\tau := \bigvee_{1 \leq i \leq k} \exists x\, \psi'_{[x \mapsto i]}$. When $\varphi = x \in X$, we let

$$\varphi'_{[x \mapsto i]} := \bigvee_{1 \leq j \leq k} \exists y\, (x, i) \sim (y, j) \wedge y \in X_j$$

For $\varphi = \mathtt{s}(x)$, we let $\varphi'_{[x \mapsto i]} := \bigwedge_{1 \leq j \leq k} \forall y\, (x, i) \sim (y, j) \implies \mathtt{s}(y, j)$, where $\mathtt{s}(y, j)$ is defined as the disjunction of all $D(y)$ where $evt(D, j) \in S_D$. We define $\varphi'_{[x \mapsto i]}$ similarly when $\varphi = \mathtt{t}(x)$. For $\varphi = x < y$ we let

$$\varphi'_{[x \mapsto i, y \mapsto j]} := \bigwedge_{1 \leq i', j' \leq k} \forall x', y'. \big((x', i') \sim (x, i) \wedge (y', j') \sim (y, j)\big) \implies x' < y'.$$

For $\varphi = x \dashrightarrow y$ we let

$$\varphi'_{[x \mapsto i, y \mapsto j]} := \bigvee_{1 \leq i' < j' \leq k} \exists z\, (z, i') \sim (x, i) \wedge (z, j') \sim (y, j).$$

Finally, when $\varphi = a(x)$, then we let $\varphi'_{[x \mapsto i]}$ be the disjunction of all $D(x)$ where $evt(D, i)$ is labelled by a. ◻

As a consequence, we obtain:

Proposition 14. *Let φ be an MSO sentence over* iiPoms*, $k \in \mathbb{N}$, and $L = \{P \in \mathsf{iiPoms}_{\leq k} \mid P \models \varphi\}{\downarrow}$. Then L is effectively regular.*

Proof. Let $K = \{P \in \mathsf{iiPoms}_{\leq k} \mid P \models \varphi\}$. By Lem. 12, $L' = \{P_1 \dots P_n \in (\Omega_{\leq k} \setminus \{\mathsf{id}_\emptyset\})^+ \mid P_1 * \dots * P_n \in K\}$ is effectively MSO^k_Ω-definable, and thus so is $L'' = \{P_1 \dots P_n \in \Omega^+_{\leq k} \mid P_1 * \dots * P_n \in K\}$. Indeed, whether $\mathsf{id}_\emptyset$ satisfies φ is decidable and so is whether $\mathsf{id}_\emptyset$ is in L''. By the standard Büchi and Kleene theorems, L'' is obtained from $\emptyset$ and $\Omega_{\leq k}$ using $\cup$, $\cdot$ and $^+$. By replacing concatenation of words by gluing composition, L is rational and thus effectively regular by Thm. 5. □

6 From HDAs to MSO

In this section we prove the second assertion of Thm. 8. The proof adapts the classical construction, encoding accepting paths of an automaton, to the case of HDAs. Our construction relies on the uniqueness of the sparse step decomposition (Lem. 2) and the MSO-definability of the relation: "an event is started/terminated before another event is started/terminated" in a sparse step decomposition (Lem. 17 below).

More formally, let $P \in \mathsf{iiPoms}$, then P admits a unique sparse step decomposition $P = P_1 * \dots * P_n$. Given $e \in P \setminus S_P$, we denote by $\mathsf{St}(e)$ the step where e is started in the decomposition, *i.e.*, the minimal i such that $e \in P_i$. For $e \in P \setminus T_P$, we similarly denote by $\mathsf{Te}(e)$ the step where e is terminated. For $x \in S_P$ we let $\mathsf{St}(x) = -\infty$ and for $x \in T_P$, $\mathsf{Te}(x) = +\infty$. Then P_i contains precisely all $e \in P$ such that $\mathsf{St}(e) \leq i \leq \mathsf{Te}(e)$, that is all events which are started before or at P_i (or never) and are terminated after or at P_i (or never). In particular, if P_i is a starter, then it starts all e such that $\mathsf{St}(e) = i$, and if it is a terminator, it terminates all e such that $\mathsf{Te}(e) = i$. Note that $\mathsf{St}(e) < \mathsf{Te}(e)$ for all $e \in P$.

Example 15. Proceeding with Ex. 13, let $w_2 = P_1 \dots P_6 = [\begin{smallmatrix} a\bullet \\ \bullet c\bullet \end{smallmatrix}][\begin{smallmatrix} \bullet a\bullet \\ \bullet c \end{smallmatrix}][\begin{smallmatrix} \bullet a\bullet \\ \bullet d\bullet \end{smallmatrix}][\begin{smallmatrix} \bullet a \\ \bullet d\bullet \end{smallmatrix}][\begin{smallmatrix} a\bullet \\ \bullet d\bullet \end{smallmatrix}][\begin{smallmatrix} \bullet a \\ \bullet d \end{smallmatrix}]$ be the sparse step decomposition of P (see also Ex. 4). We have $\mathsf{St}(e_3) = -\infty, \mathsf{St}(e_1) = 1, \mathsf{St}(e_4) = 3$ and $\mathsf{St}(e_2) = 5$. Also, $\mathsf{Te}(e_3) = 2, \mathsf{Te}(e_1) = 4$ and $\mathsf{Te}(e_2) = \mathsf{Te}(e_4) = 6$. Further, P_1 contains e_1 since $\mathsf{St}(e_1) = 1$ and e_3 because $\mathsf{St}(e_3) \leq 1 \leq \mathsf{Te}(e_3)$; P_4 contains e_1 since $\mathsf{Te}(e_1) = 4$ and e_4 because $\mathsf{St}(e_4) \leq 4 \leq \mathsf{Te}(e_4)$.

The example above will be continued in Ex. 19 at the end of this section. It will be pertinent in particular for the next lemma describing the existence of an accepting path inducing a sparse step decomposition as the existence of labellings $\rho_\nearrow$ and $\rho_\searrow$ mapping each started or terminated event of P to the upstep or downstep of the HDA performing it.

Lemma 16. *Let $\mathcal{H}$ be an HDA and $P \in \mathsf{iiPoms} \setminus \mathsf{Id}$ whose sparse step decomposition is $P_1 * \dots * P_n$. We have $P \in L(\mathcal{H})$ if and only if there exist $\rho_\nearrow \colon P \setminus S_P \to \mathsf{ups}(\mathcal{H})$ and $\rho_\searrow \colon P \setminus T_P \to \mathsf{downs}(\mathcal{H})$ such that, for all $e_1, e_2 \in P$:*

1. *if* $\mathsf{St}(e_1) = \mathsf{St}(e_2)$ *then* $\rho_{\nearrow}(e_1) = \rho_{\nearrow}(e_2)$;
2. *if* $\mathsf{Te}(e_1) = \mathsf{Te}(e_2)$ *then* $\rho_{\searrow}(e_1) = \rho_{\searrow}(e_2)$;
3. *if* $\mathsf{St}(e_2) = \mathsf{Te}(e_1) + 1$ *then* $\mathsf{src}(\rho_{\nearrow}(e_2)) = \mathsf{tgt}(\rho_{\searrow}(e_1))$;
4. *if* $\mathsf{Te}(e_2) = \mathsf{St}(e_1) + 1$ *then* $\mathsf{src}(\rho_{\searrow}(e_2)) = \mathsf{tgt}(\rho_{\nearrow}(e_1))$;
5. *if* $\rho_{\nearrow}(e_1) = (p, \nearrow^A, q)$ *then*

$$\begin{aligned} A =& (U = \{e \mid \mathsf{St}(e) = \mathsf{St}(e_1)\}, \dashrightarrow_{P_{\restriction U}}, \lambda_{P_{\restriction U}}), \\ \mathsf{ev}(q) =& (V = \{e \mid \mathsf{St}(e) \leq \mathsf{St}(e_1) < \mathsf{Te}(e)\}, \dashrightarrow_{P_{\restriction V}}, \lambda_{P_{\restriction V}}); \end{aligned}$$

6. *if* $\rho_{\searrow}(e_1) = (p, \searrow_A, q)$ *then*

$$\begin{aligned} A =& (U = \{e \mid \mathsf{Te}(e) = \mathsf{Te}(e_1)\}, \dashrightarrow_{P_{\restriction U}}, \lambda_{P_{\restriction U}}), \\ \mathsf{ev}(p) =& (V = \{e \mid \mathsf{St}(e) < \mathsf{Te}(e_1) \leq \mathsf{Te}(e)\}, \dashrightarrow_{P_{\restriction V}}, \lambda_{P_{\restriction V}}); \end{aligned}$$

7. *if* $\mathsf{St}(e_1) = 1$ *then* $\mathsf{src}(\rho_{\nearrow}(e_1)) \in \bot_H$;
8. *if* $\mathsf{Te}(e_1) = 1$ *then* $\mathsf{src}(\rho_{\searrow}(e_1)) \in \bot_H$;
9. *if* $\mathsf{St}(e_1) = n$ *then* $\mathsf{tgt}(\rho_{\nearrow}(e_1)) \in \top_H$;
10. *if* $\mathsf{Te}(e_1) = n$ *then* $\mathsf{tgt}(\rho_{\searrow}(e_1)) \in \top_H$.

As $P \notin \mathsf{Id}$, $\rho_{\nearrow}$ or $\rho_{\searrow}$ must be defined for at least one element of P above.

Our goal is to show that the conditions given by Lem. 16 can be expressed in MSO. We want to define a formula $\exists X_1 \dots \exists X_m . \exists Y_1 \dots \exists Y_n . \varphi$ with one X_i (resp. Y_j) for each upstep (resp. downstep) of the HDA. Intuitively, each X_i (Y_j) will contain all the events started (terminated) by performing the corresponding upstep (downstep). The sentence φ expresses that each event belongs to exactly one X_i (unless it is a source, in which case it belongs to none) and one Y_i (unless it is a target), and that the resulting labellings $\rho_{\nearrow}$ and $\rho_{\searrow}$ satisfy the conditions of the lemma. Hence, identity events do not belong to any X_i or Y_j. Nevertheless, conditions 5 and 6 ensure that they are consistent with the encoded path.

Let us first prove that the relations used in Lem. 16 are MSO-definable.

Lemma 17. *For* $f, g \in \{\mathsf{St}, \mathsf{Te}\}$ *and* $\bowtie \in \{=, <, >\}$, *the relations* $f(x) \bowtie g(y)$, $\min(f)$ *and* $\max(f)$ *are MSO-definable.*

Proof. We first define $\mathsf{Te}(x) < \mathsf{St}(y)$ as the formula $x < y$, together with $\mathsf{St}(x) < \mathsf{Te}(y) := \neg(\mathsf{Te}(y) < \mathsf{St}(x))$. Because starters and terminators alternate in the sparse step decomposition, we can then let

$$\begin{aligned} \mathsf{St}(x) < \mathsf{St}(y) &:= \exists z . \mathsf{St}(x) < \mathsf{Te}(z) \wedge \mathsf{Te}(z) < \mathsf{St}(y), \\ \mathsf{St}(x) = \mathsf{St}(y) &:= \neg(\mathsf{St}(x) < \mathsf{St}(y)) \wedge \neg(\mathsf{St}(y) < \mathsf{St}(x)) \wedge \neg\mathsf{s}(x) \wedge \neg\mathsf{s}(y) \\ \min(\mathsf{St}(x)) &:= \neg\mathsf{s}(x) \wedge \neg\exists y . \mathsf{Te}(y) < \mathsf{St}(x) \\ \max(\mathsf{Te}(x)) &:= \neg\mathsf{t}(x) \wedge \neg\exists y . \mathsf{St}(y) > \mathsf{Te}(x) . \end{aligned}$$

The other formulas are defined similarly. □

We can also define $\mathsf{St}(y) = \mathsf{Te}(x) + 1$ and $\mathsf{Te}(y) = \mathsf{St}(x) + 1$ using standard techniques. Observe that $\mathsf{Te}(x) < \mathsf{St}(y)$ implies $\neg t(x) \wedge \neg\mathsf{s}(y)$, given that the end of the x-event precedes the beginning of the y-event. As a consequence $\mathsf{St}(x) < \mathsf{St}(y)$ implies $\neg\mathsf{s}(y)$. On the other hand $\mathsf{St}(x) < \mathsf{Te}(y)$ holds in particular when x or y are interpreted as identities.

Proposition 18. *Given an HDA $\mathcal{H}$, one can effectively construct an MSO sentence φ such that $L(\mathcal{H}) = \{P \in \mathsf{iiPoms} \mid P \models \varphi\}$.*

Proof. We define

$$\begin{aligned}\varphi := (\exists x.\, \neg\mathsf{s}(x) \vee \neg\mathsf{t}(x)) \implies \exists X_1, \dots, X_m.\, \exists Y_1, \dots, Y_n.\, \textstyle\bigwedge_{i=0,\dots,10}\varphi_i \\ \wedge\, (\forall y.\, \mathsf{s}(y) \wedge \mathsf{t}(y)) \implies \textstyle\bigvee_{\substack{p\in\bot_H\cap\top_H \\ \mathsf{ev}(p)\neq\emptyset}} \exists y_1, \dots, y_{|\mathsf{ev}(p)|}.\mathsf{ev}(p)(y_1, \dots, y_{|\mathsf{ev}(p)|}).\end{aligned}$$

where φ_0 checks that the X_i's and Y_i's define labellings $\rho_\nearrow$ and $\rho_\searrow$ as in Lem. 16, that is, each event belongs to at most one X_i (is associated with at most one upstep) and one Y_i, and to no X_i iff it is a source and to no Y_i iff it is a target. The other formulas φ_i check condition i of Lem. 16. The second line of φ is satisfied by all non-empty identities accepted by $\mathcal{H}$. Thus $L(\varphi) = L(\mathcal{H}) \setminus \{\mathsf{id}_\emptyset\}$. If $\mathsf{id}_\emptyset \in L(\mathcal{H})$ then $L(\mathcal{H}) = L(\varphi \vee \neg\exists x.\, \mathtt{true})$. □

Example 19. Let $\mathcal{H}$ be the HDA of Fig. 3 and P the ipomset of Fig. 4. Recall that P is accepted by $\mathcal{H}$ by (among others) the path

$$t_3 \nearrow^a q_1 \searrow_c t_2 \nearrow^d q_2 \searrow_a t_8 \nearrow^a q_3 \searrow_{ad} v_8$$

which induces the sparse step decomposition

$$\left[\begin{smallmatrix} a\bullet \\ \bullet c\bullet \end{smallmatrix}\right] * \left[\begin{smallmatrix} \bullet a\bullet \\ \bullet c \end{smallmatrix}\right] * \left[\begin{smallmatrix} \bullet a\bullet \\ d\bullet \end{smallmatrix}\right] * \left[\begin{smallmatrix} \bullet a \\ \bullet d\bullet \end{smallmatrix}\right] * \left[\begin{smallmatrix} a\bullet \\ \bullet d\bullet \end{smallmatrix}\right] * \left[\begin{smallmatrix} \bullet a \\ \bullet d \end{smallmatrix}\right].$$

Let

$$\begin{gathered}\rho_\nearrow(e_1) = t_3 \nearrow^a q_1, \quad \rho_\nearrow(e_2) = t_8 \nearrow^a q_3, \quad \rho_\nearrow(e_4) = t_2 \nearrow^d q_2, \\ \rho_\searrow(e_1) = q_2 \searrow_a t_8, \quad \rho_\searrow(e_2) = \rho_\searrow(e_4) = q_3 \searrow_{ad} v_8, \quad \rho_\searrow(e_3) = q_1 \searrow_c t_2.\end{gathered}$$

These definitions of $\rho_\nearrow$ and $\rho_\searrow$ satisfy the conditions of Lem. 16. Likewise, let $\varphi_\mathcal{H}$ be the MSO sentence built from $\mathcal{H}$ as in Prop. 18. Then $P \models \varphi_\mathcal{H}$, since the following interpretation satisfies $\bigwedge_{i=0,\dots,10}\varphi_i$:

$$\begin{gathered}X_{t_3\nearrow^a q_1} = \{e_1\}, \quad X_{t_2\nearrow^d q_2} = \{e_4\}, \quad X_{t_8\nearrow^a q_3} = \{e_2\}, \\ Y_{q_1\searrow_c t_2} = \{e_3\}, \quad Y_{q_2\searrow_a t_8} = \{e_1\}, \quad Y_{q_3\searrow_{ad} v_8} = \{e_2, e_4\},\end{gathered}$$

and the other X_u, Y_d for $u \in \mathsf{ups}(\mathcal{H})$ and $d \in \mathsf{downs}(\mathcal{H})$ are empty. Note that $\rho_\nearrow$ is not defined for e_3 since it is a source for P. For the same reason e_3 does not belong to any interpretation of X_u for any $u \in \mathsf{ups}(\mathcal{H})$.

7 Conclusion

This paper enriches the language theory of higher-dimensional automata with a Büchi-Elgot-Trakhtenbrot-like theorem. We have shown that the subsumption closures of MSO-definable subsets of $\mathsf{iiPoms}_{\le k}$ are regular and that regular languages of HDAs are MSO-definable, both with effective constructions. Also, the MSO theory of $\mathsf{iiPoms}_{\le k}$ and the MSO model-checking for HDAs are decidable.

Theorem 8 induces also a construction, for an MSO sentence φ over $\mathsf{iiPoms}_{\leq k}$, of $\varphi\downarrow$ such that $L(\varphi\downarrow) = L(\varphi)\downarrow$. This property fails when we consider non-interval pomsets. However, the construction of $\varphi\downarrow$ is not efficient, as the current workflow is to transform φ to an HDA and then get $\varphi\downarrow$. We are wondering whether a more direct construction is possible.

Our work could be continued by considering logics weaker than MSO. For example, the study of the expressive power of first order logic over $\mathsf{iiPoms}_{\leq k}$ would be useful for model-checking purposes. In this regard, another operational model that would naturally arise is a class of ω-HDAs: HDAs over infinite ipomsets.

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