A Vectorial Self-Dual Morphological Filter based on Total Variation Minimization

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Abstract. We present a vectorial self dual morphological filter. Contrary to many methods, our approach does not require the use of an ordering on vectors. It relies on the minimization of the total variation with L^1 norm as data fidelity on each channel. We further constraint this minimization in order not to create new values. It is shown that this minimization yields a self-dual and contrast invariant filter. Although the above minimization is not a convex problem, we propose an algorithm which computes a global minimizer. This algorithm relies on minimum cost cut-based optimizations.

1 Introduction

It is well known that morphological filters on gray-scale images feature the property of being invariant with respect to any change of contrast and that they do not create new gray levels [11]. One of the main issue for mathematical morphology is its extension to the vectorial case. The difficulty arises because a total order between elements is required using the classical approach of lattice theory [19]. Extensions of this approach to the vectorial case have been tackled using vector ranking concepts [10, 20]. In this paper, we propose a new, self dual morphological filter for vectorial images, based on total variation minimization. Our approach does not require any order relation between vectors.

A lot of work has been devoted to the design of morphological operators for *color* images as a specific case of vectorial images. Main approaches consist in choosing a suitable color space representation and defining an ordering relationship [8, 13]. In [5], Chambolle proposes a definition for contrast invariance of operators on colors. This definition is considered by Caselles *et al.* in [4] who present a morphological operator on color images.

Vector median filters are another approach for vector filtering, originally introduced by Astola *et al.* in [1]. The process consists of replacing the pixel value by the median of the pixels contained in a window around it. The median is defined as the value that minimizes the L^1-norm between all pixels in a window.

This method has been used specially for noise filtering [14–16]. In [4], Caselles *et al.* connect vector median filters, morphological operators and partial differential equations. They consider a lexicographic order to obtain these connections. Complementary results on these links for the scalar case can be found in [11].

In [6], the author deals with the scalar case and shows that minimization of the total variation (TV) under the L^1-norm as data fidelity yields a morphological filter. This model will be referred to as $L^1 + TV$. Assume that an observed image v is defined on Ω and takes values in $\mathbb{R}$. For sake of clarity, we assume here that Ω is a rectangle of $\mathbb{R}^2$ although all the results presented in the paper apply for any convex set in any dimension. The energy associated to the model $L^1 + TV$ is expressed as follows:

$$E(u) = \int_\Omega |u(x) - v(x)| dx + \beta \int_\Omega |\nabla u| \ , \quad (1)$$

where the last term is the TV of u weighted by a non-negative coefficient β. Note that the gradient is taken in the distributional sense. An efficient algorithm is proposed in [7] to perform an exact minimization of (1), i.e, it provides a global minimizer. In [2], Blomgren *et al.* propose some extensions of the total variation definition to the vectorial case. They study them for image restoration purposes. However, no relation with mathematical morphology is introduced.

The contributions of this paper are the following. We propose a morphological filter based on the minimization of TV. Contrary to many previous approaches, our method does not require any order relationship between vectors. Our approach relies on extending the energy (1) to the vectorial case by simply applying the $L^1 + TV$ model on each channel. We further constrain the energy such that no new value is created. We show that this filter is a morphological one. Contrary to the minimization of (1), the problem is *not anymore convex*. We thus propose an algorithm which provides an exact minimizer for this new non-convex functional. This algorithm relies on minimum-cost cut ones. To our knowledge, these results are new.

The structure of the paper is as follows. Section 2 is dedicated to the presentation of the proposed approach for the design of our vectorial morphological filter. We present an algorithm to perform the minimization in Section 3, along with some results.

2 Vectorial mathematical morphology

In this section we briefly review the $L^1 + TV$ model. Then we show how to generalize the approach to the vectorial case. We define a continous change of contrast as follows [12]: any continuous non-decreasing function on $\mathbb{R}$ is called a continuous change of contrast. The following theorem is proved in [6] and in appendix A.

Theorem 1 *Let v be an observed image and g be a continuous change of contrast. Assume u to be a global minimizer of $E_v(\cdot)$. Then $g(u)$ is a global minimizer of $E_{g(v)}(\cdot)$. Besides, $-u$ minimizes $E_{-v}(\cdot)$.*

We now deal with the vectorial case. From now on, we consider vectorial images, $u = (u^1, ...u^N)$, defined on Ω which take values into $\mathbb{R}^N$. We define the L^1-norm $\|u\|_{L^1}$ for a vectorial image u as the sum of L^1-norms on each channel, i.e: $\|u\|_{L^1} = \sum_{i=1}^N \int_\Omega |u^i(x)|dx$. We extend the total variation $\overrightarrow{tv}(u)$ of a vectorial image u in the same way, i.e: $\overrightarrow{tv}(u) = \sum_{i=1}^N \int_\Omega |\nabla u^i|$. A straightforward extension of the scalar model $L^1 + TV$ to the vectorial case consists in applying *independently* $L^1 + TV$ on each channel, i.e:

$$E(u) = \|u - v\|_1 + \beta\, \overrightarrow{tv}(u) \ . \tag{2}$$

However, it is easily seen that if no constraint is added, minimization of this energy yields a minimizer which has new vectorial values. Consequently it breaks the morphological property. Thus we add a constraint to ensure that a global minimizer does not have new values. Let us denote by $\mathcal{C}$ the set of all vectorial values appearing in the observed image v. Our goal is to find a global minimizer of the following problem:

$$\text{(P)} \begin{cases} \underset{u}{\text{arginf}}\ \|u - v\|_1 + \beta\, \overrightarrow{tv}(u) \\ \text{s. t. } \forall x \in \Omega \ \ u(x) \in \mathcal{C} \end{cases}$$

As one can see, our extension to the vectorial case reduces to the classical $L^1 + TV$ model when images are scalar. We now give our definition for a vectorial change of contrast.

Definition 1 *Any continuous function $g : \mathbb{R}^N \mapsto \mathbb{R}^N$ is called a vectorial continous change of contrast if and only of its restriction to any canonic axis is a continuous change of contrast.*

Then it is easily seen that problem (P) defines a morphological filter according to our vectorial change of contrast definition (1).

Note that although minimization of the scalar model $L^1 + TV$ defined by equation (1) is a convex problem, *it is no longer the case for problem* (P). Indeed, the objective function is still convex, but the constraint is not. An algorithm for computing an exact solution of the non-convex problem (P) is given in the next section.

3 Minimization Algorithm and Results

In this section, we present an algorithm which computes a global optimizer for a discrete version of problem (P) along with some results. In the following we assume that images are defined on a discrete lattice S and take values in $\mathcal{C}$. We denote by u_s the value taken by the image u at the site $s \in S$. Two neighboring sites s and t are denoted $s \sim t$. The discrete version of energy (2) is thus as follows:

$$E(u) = \sum_{i=1}^N \sum_s |u_s^i - v_s^i| + \beta \sum_{(s \sim t)} |u_s^i - u_t^i| \ .$$

Now we present our algorithm for optimizing energy E.

```
Start with a labeling u such that ∀s u_s ∈ C
do
   success ← false
   forall α ∈ C
        u' = argmin_û E(û) where û is an α−expansion of u

        if E(u') < E(u)
             u ← u'
             success ← true
while success ≠ false
```

Fig. 1. Pseudo-code for our minimization algorithm.

3.1 Minimimum Cost Cut Based Minimization

Our algorithm relies on the α-expansion moves algorithm proposed by Boykov *et al.* in [3]. An α-expansion move from a current labeling is defined as follows: given a value α, every pixel can either keep its current value or take α. We are interested in finding the optimal α-expansion move from a current labeling which minimizes the energy. Originally, this method is devoted to the approximation of non-convex Markovian energy [3].

In order to solve problem (P), we iterate optimal α-expansion moves. At each iteration we perform an optimal α-expansion move where α belongs to the set of observed values $\mathcal{C}$. The traversal on $\mathcal{C}$ stops when no α-expansion can furthermore decrease the energy. This algorithm is presented in Figure 1. We now prove, in the following proposition, that this algorithm provides a global minimizer for the non-convex problem (P).

Theorem 1 *Let u be an image such that*

$$E(u) > \inf_{u'} E(u') \ .$$

Then, there exists u^α which is within one $\alpha-$expansion move of u, such that

$$E(u) > E(u^\alpha) \ .$$

Proof: Before giving to the proof, we recall that for a one dimensional discrete convex function $f : \mathbb{Z} \mapsto \mathbb{R}$, the following inequality holds [17] :

$$\forall x \, \forall y \forall d \mid (y \geq x) \wedge (0 \leq d \leq (y-x)), \quad f(x) + f(y) \geq f(x+d) + f(y-d) \ . \quad (3)$$

Let $\hat{u}$ be a global minimizer of E, i.e., $E(\hat{u}) = \inf_{u'} E(u')$ Given a value $\alpha \in \mathcal{C}$, we define an image δ as follows:

$$\forall s \ \ \delta_s = \begin{cases} \alpha - u_s & \text{if } \alpha \in [\![u_s, \hat{u}_s]\!] \text{ or } \alpha \in [\![\hat{u}_s, u_s]\!] \ , \\ 0 & \text{else.} \end{cases} \quad (4)$$

• We first prove the following inequality:

$$E(u) + E(\hat{u}) \geq E(u + \delta) + E(\hat{u} - \delta) \ . \tag{5}$$

We show it for the data fidelity and regularisation terms independently.
- Data fidelity terms: Since the absolute value is a convex function, we have the following inequality (obtained using equation (3)) for all data fidelity terms:

$$|u_s - v_s| + |\hat{u}_s - v_s| \geq |u_s + \delta_s - v_s| + |\hat{u}_s - \delta_s - v_s| \ .$$

This concludes the proof for the first case.
- *A priori* terms: Let $X_{st} = u_s - u_t$, $Y_{xt} = \hat{u}_s - \hat{u}_t$ and $D_{st} = \delta_s - \delta_t$. Assume that $X_{st} \leq Y_{st}$. Thus we have by definition of δ (equation (4)), $X_{st} \leq X_{st} + D_{st} \leq Y_{st}$. Applying inequality (3) we have the desired results. The case $X_{st} > Y_{st}$ is similar to this one.
• Let us denote by $\mathcal{M}$ the set of global minimizers of $E(\cdot)$. Let us define the norm $\|u\|_1$ on an image u as $\|u\|_1 = \sum_s |u_s|$. Let us define $u^\star$ as follows:

$$u^\star = \underset{u' \in \mathcal{M}}{\operatorname{argmin}} \ \|u' - u\|_1 \ . \tag{6}$$

From equation (5), we have $E(u) - E(u + \delta) \geq E(\hat{u} - \delta) - E(\hat{u}) \geq 0$, since $\hat{u}$ is a global minimizer. If the inequality is strict then the proof is finished. If this is not the case then we have $E(\hat{u} - \delta) = E(\hat{u})$. However, it is easy to show that $\|u + \delta - \hat{u}\|_1 < \|u - \hat{u}\|_1$. This is in contradiction with the definition of $\hat{u}$ (equation (4)). This concludes the proof. □

We use a minimum cost cut technique to find the optimal $\alpha-$expansion move that must be done in order to decrease the energy [3]. This minimum cost cut is computed on a weighted graph corresponding to the energy associated with an $\alpha-$expansion.

3.2 Results

We present some results on color images. Since, in this paper, we focus on demonstrating the effectiveness of our filter, we applied our model to the RGB space. We are aware that many other color spaces are available [18] and we are currently studying more suitable spaces. Figure 2 presents some results on the image *hand*. We note that the higher the coefficient β, the more the image is simplified. Details of the texture are removed while the geometry is kept. Moreover, the colors of the background of the hand do not merge. Finally, the color of the ring is well preserved. This result gives a very good initialization for a segmentation process. In [21], the authors perform the minimization of the $L^1 + TV$ model in order to decompose the image into two parts: the first one contains the geometry while the second one contains the textures. Figure 2 depicts the result of such a decomposition using our filter. We performed a change of contrast on gray levels to enhance the content of the textured image. Note how fine the decomposition is.

4 Conclusion

In this paper we have proposed a new morphological filter for vectorial images. The main feature of this filter is that it does not involve any ordering between elements. We have also presented an algorithm to perform the filtering. Many opportunities for future work are considered. First, a faster algorithm is currently under investigation. The special case of color images must be handled by applying this filter on color spaces other than RGB [18], such as Lab. All these extensions will be presented in a forthcoming paper.

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A Proof of Theorem 1

We first introduce the notion of level sets of an image and give two lemma.

Definition 1 *The lower level sets of an image u, referred to as u^λ are defined as follows:*

$$u^\lambda(x) = \mathbb{1}_{u(x)\leq\lambda} \ .$$

Before we give the proof of Theorem 1, we give a lemma proved in [11], which stipulates that after a continuous change of contrast g, the level sets of an image $g(v)$ are some level sets of the image v.

Lemma 1 *Assume g to be a continuous change of contrast and u an image defined on Ω. The following holds: $\forall\lambda\,\exists\mu\ \ (g(u))^\lambda = u^\mu$.*

Lemma 2 *Let us denote χ_A the characteristic function of the set A. The energy $E_v(u)$ rewrites as follows for almost all λ:*

$$E_v(u) = \int_{\mathbb{R}} E_v^\lambda(u^\lambda, v^\lambda) d\lambda \ , \ \ where$$
$$E_v^\lambda(u^\lambda, v^\lambda) = \int_\Omega \left(\beta\left|\nabla\chi_{u^\lambda}\right| + \left|u^\lambda(x) - v^\lambda(x)\right| dx\right) \ .$$

Proof: The fidelity term rewrites as follows:

$$|u(x) - v(x)| = \int_{\mathbb{R}} \left|u^\lambda(x) - v^\lambda(x)\right| d\lambda \ .$$

The co-area formula [9] states that for any function which belongs to the space of functions of bounded variation, we have: $\int_\Omega |\nabla u| = \int_{\mathbb{R}}\int_\Omega |\nabla\chi_{u^\lambda}|\, d\lambda$, for almost all λ. This concludes the proof of this lemma. □

Proof of theorem 1: First, we show that $L1 + TV$ is invariant with respect to any change of contrast. It is sufficient to prove that for any level λ, a minimizer for $g(v)^\lambda$ is $g(u)^\lambda$. Using lemma 1, there exists μ such that $v^\mu = g(v)^\lambda$. A minimizer of $E_v^\mu(\cdot, v^\mu)$ is u^μ. Thus, u^μ is a minimizer of $E_v^\mu(\cdot, g(v)^\lambda)$. And we have $u^\mu = g(u)^\lambda$.

Self dual invariance is easily obtained. It is enough to note that $\int_\Omega |\nabla u| = \int_\Omega |\nabla(-u)|$ and that $\int_\Omega |u(x) - v(x)| dx = \int_\Omega |(-u(x)) - (-v(x))| dx$. The conclusion is straightforward. □

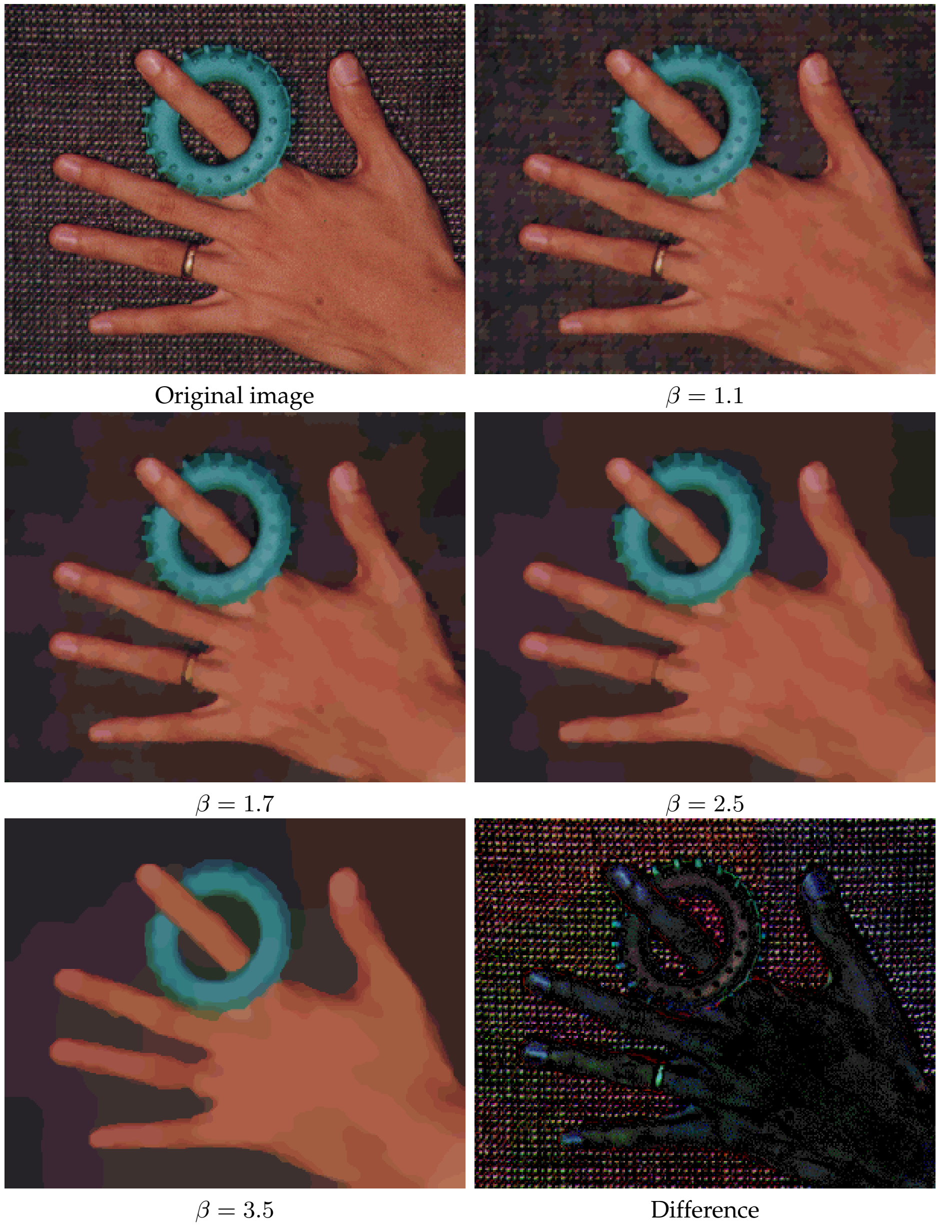

Fig. 2. Minimizers of problem (P) for different regularization coefficients for the image *hand*. The last image *difference* is the difference between the original image and the one for $\beta = 3.5$ (Note that we applied a change of contrast on the gray levels to enhance the colors).