Bisimulations and Logics for Higher-Dimensional Automata

Safa Zouari[1], Krzysztof Ziemiański[2], and Uli Fahrenberg[3]

[1] Norwegian University of Science and Technology, Gjøvik, Norway
safa.zouari@ntnu.no
[2] University of Warsaw, Poland ziemians@mimuw.edu.pl
[3] EPITA Research Lab, Rennes, France
uli@lrde.epita.fr

Abstract. Higher-dimensional automata (HDAs) are models of non-interleaving concurrency for analyzing concurrent systems. There is a rich literature that deals with bisimulations for concurrent systems, and some of them have been extended to HDAs. However, no logical characterizations of these relations are currently available for HDAs.
In this work, we address this gap by introducing Ipomset modal logic, a Hennessy-Milner type logic over HDAs, and show that it characterizes Path-bisimulation, a variant of the standard ST-bisimulation. We also define a notion of Cell-bisimulation, using the open-maps framework of Joyal, Nielsen, and Winskel, and establish the relationship between these bisimulations (and also their "strong" variants, which take restrictions into account). In our work, we rely on a categorical definition of HDAs as presheaves over concurrency lists and on track objects.

Keywords: Higher Dimensional Automaton · Ipomset Modal Logic · Hennessy-Milner logic · Bisimulation · Open map · Pomset

1 Introduction

Higher-Dimensional Automata (HDAs), introduced by Vaughan Pratt [25] and Rob van Glabbeek [14], are a powerful model for non-interleaving concurrency. Van Glabbeek [14] places HDAs at the top of a hierarchy of concurrency models, demonstrating how other concurrency models, such as Petri nets [22], configuration structures [29], asynchronous systems [7,28], and event structures [31,32], can be incorporated into HDAs.

As its name implies, a Higher-Dimensional Automaton consists of a collection of n-dimensional hypercubes or n-cells connected via source and target maps. The well-known automata or labeled transition systems are 1-dimensional HDAs. However, HDAs allow for more expressive modeling of concurrent and distributed systems. For example, the concurrent execution of two events a and b can be modeled by a square labeled as in Fig. 2, while an empty square represents mutual exclusion. Analogously, a filled-in 3-dimensional cube in an HDA can represent three events a_1, a_2, and a_3 that execute concurrently, while when considering a hollow cube, each 2-dimensional face models $a_i \parallel a_j$ for $1 \leq i \neq j \leq 3$. See Fig. 1.

A *higher-dimensional automaton* is a *precubical set* together with an *initial cell* and a set of *final cells*. Like a simplicial set, a *precubical set* is constructed by systematically gluing hypercubes. Formally, it is a graded set $X = \bigcup_{n \in \mathbf{N}} X_n$, where X_n represents the set of n-*cells*, with face maps δ^0 resp δ^1 defining the mapping of an n-cell to its lower resp. upper face. Each n-cell is associated with a linearly ordered and labeled set V of length n. From a concurrency point of view, such a cell models a list V of n active events. A lower face of a cell $x_n \in X_n$ is a cell $\delta^0_{V \setminus U}(x)$ that has $U \subseteq V$ as active events. For example, in Fig.3, the square has active *events* $[ab]$. Its faces have active events $[a]$ and $[b]$, respectively. In Section 2, we make this precise by defining *precubical sets* as presheaves over a category of linearly ordered sets with appropriate morphisms [9,10].

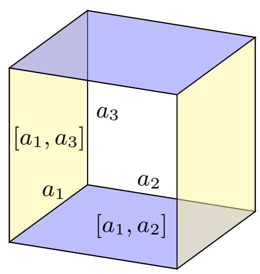


Fig. 1: How HDA models concurrency: The filled-in cube models the events $[a_1, a_2, a_3]$. The 2-dimensional faces with the same color model the same 2 events. The uncolored faces model the events $[a_2, a_3]$.

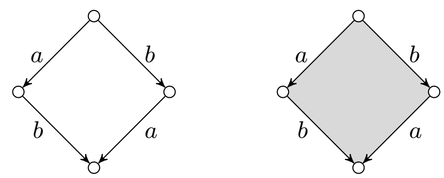


Fig. 2: HDA models distinguishing interleaving $a.b + b.a$ (left) from non-interleaving concurrency $a \parallel b$ (right).

In addition to concurrency models, equivalence relations should also be considered when describing concurrent systems. Various notions of equivalence have been suggested in studies [26,15,16,20,13,17], guided by considerations of the critical aspects of system behavior within a specific context and the elements from which to abstract. Parallel to behavioral equivalences, modal logic is a useful formalism for specifying and verifying properties of concurrent systems [1,24,3,6].

Characterization of bisimulation in terms of Hennessy-Milner logic (HML) provides additional confidence in both approaches. Two finitely branching systems are bisimilar iff they satisfy the same logical assertions. The literature focusing on logical characterization includes the Van Glabbeek spectrum [13] for sequential processes and [4,21,8,27,23,5] for concurrent systems. Some of these behavioral equivalences have been extended to HDAs. Among them are hereditary history-preserving bisimulations (hh-bisimulation) and ST-bisimulations

[14]. However, to the best of our knowledge, their logical counterparts have not been investigated for HDA.

This paper presents a variant of HML interpreted over HDAs, namely *Ipomset Modal Logic* (IPML). The original HML in the interleaving setting [17] contains negation ($\neg$), conjunction ($\wedge$), a formula $\top$ that always holds, and a diamond modality $\langle a \rangle F$, which says that it is possible to perform an action labeled by a and reach a state that satisfies F. Unlike the standard HML, IPML considers both sequential and concurrent computations. Thus, it differs from the standard HML within the diamond modality, so it becomes $\langle P \rangle F$ where P is an *interval pomset with interfaces* (interval ipomset). It is interpreted over a *path* α, and says that there is a path β that recognizes P and extends α to a path (concatenation of α and β) that satisfies F. For example, in Fig. 2, the formula $\langle (a \longrightarrow b) \rangle \top \vee \langle (b \longrightarrow a) \rangle \top$, which stands for mutual exclusion, holds at the top edge of both squares. However, formula $\langle [\begin{smallmatrix} a \\ b \end{smallmatrix}] \rangle \top$ holds only in the upper corner of the filled-in square (on the right). The latter formula shows that our logic is powerful enough to distinguish interleaving from true concurrency.

Pomsets were first introduced by Winkowski [30]. *Interval orders*, a subclass of *pomsets*, have been introduced by Fishburn [12]. Then, they have been equipped with *interfaces* [9], facilitating the definition of the *gluing composition* of HDA languages. A computational run in an HDA is modeled by a *path*, a sequence of *cells*. Each two consecutive *cells* are related by a *source* or a *target map*. The observable contents of a path α are described by $\mathsf{ev}(\alpha)$, an *interval pomset with interfaces*. To define *cell-bisimulation* and IPML over HDAs, we

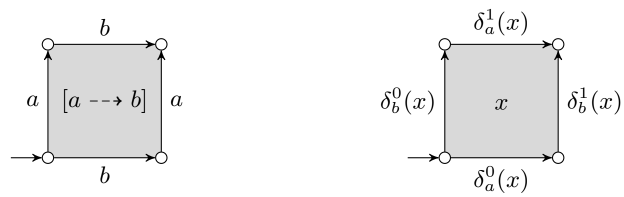


Fig. 3: Two-dimensional cell with its faces. The labels of each cell are shown on the left.

employ the notion of open map bisimulation [19]. This approach requires a category of models $\mathbf{M}$ (the category of HDAs in our case) and a path category $\mathbf{T}$ (the category of *track objects* in our case), a subcategory of $\mathbf{M}$ that we call the HDA-path category. Track objects have originally been introduced in [9] to define languages of HDAs. They form a subcategory of $\mathbf{M}$. A *track object* is a particular HDA that can be constructed from a given interval ipomset P, denoted $\square^P$. Intuitively, for a given path π labeled with an interval ipomset P, the track object $\square^P$ is the smallest HDA containing π. We show that the resulting logic characterizes path-bisimulation [9], a variant of ST-bisimulation [14]. However, its extension, equipped with backward modality characterizes the strong path-bisimulation. Finally, we finish this paper with a hierarchy of the equivalence relations encountered in the paper. This is summarized in Fig. 4.

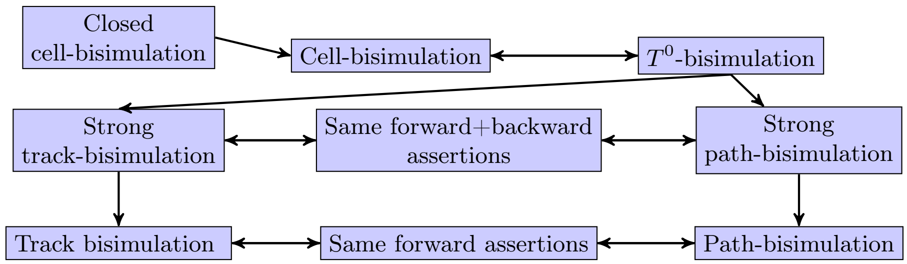


Fig. 4: Hierarchy of notions of equivalence.

Other contributions, that deepen the understanding of mathematical structures and may be of independent interest, include establishing a relation between track objects and interval ipomsets in Th. 19, a relation between the notion of paths of van Glabbeek [14] and tracks of [9] in Th. 35, and a generalization of the Yoneda lemma to interval ipomsets in Prop. 33. All proofs are available in a separate appendix.

2 Higher Dimensional Automata

We review the definition of Higher Dimensional Automata. We rely on a categorical approach proposed and studied in [9,10], where an HDA is defined as a specific precubical set. To define precubical sets as presheaves over the labeled precube category $\square$, we introduce conclists and conclist maps, which are the objects and the morphisms of $\square$. We restrict our study to finitely branching HDAs.

Definition 1. *A* concurrency list *or* conclist *is a tuple* $(U, \dashrightarrow, \lambda)$*, where* U *is a finite set totally ordered by the strict order* $\dashrightarrow$ *and* $\lambda : U \to \Sigma$ *is a labeling map. Elements of* U *will be called* events.

Definition 2. *A* conclist-map *from a conclist* U *to* T *is a pair* (f, ε) *such that:*
- $f : U \to T$ *is a label and order-preserving function;*
- $\varepsilon : T \to \{0, \lrcorner, 1\}$ *is a function such that* $\varepsilon^{-1}(\lrcorner) = f(U)$.

The composition of morphisms $(f, \varepsilon) : U \to T$ *and* $(g, \zeta) : T \to V$ *is* $(g, \zeta) \circ (f, \varepsilon) = (g \circ f, \eta)$*, where*

$$\eta(u) = \begin{cases} \varepsilon(g^{-1}(u)) & \text{for } u \in g(T), \\ \zeta(u) & \text{otherwise.} \end{cases}$$

Let $\square$ *be the category of conclists and conclist maps. We write* $U \simeq V$ *for isomorphic conclists; if two conclists are isomorphic, then the isomorphism between them is unique.*

For a conclist map $(f, \varepsilon) : U \to V$, since the order $\dashrightarrow$ is total, f is an injective function, which is determined by $V \setminus f(U) = V \setminus \varepsilon^{-1}(\lrcorner\!\ulcorner)$, and hence by ε. For instance, the identity morphism $\mathbf{id}_V^{\square} : V \to V$ is uniquely determined by ε_V where $\varepsilon_V(v) = \lrcorner\!\ulcorner$ for all $v \in V$. Intuitively the map f injects the list of events of U into the list of events of V, while the map ε guarantees that the events of U are active in V by giving them the value $\lrcorner\!\ulcorner$, and specifies the state of the remaining events by giving them the value 0 if they are not yet started and 1 if they are terminated.

Notation A morphism $(f, \varepsilon) : U \to V$ might be denoted $d_{A,B} : U \to V$ where $A = \varepsilon^{-1}(0)$ and $B = \varepsilon^{-1}(1)$. Such a morphism is usually called a coface map [10]. We write d_A^0 for $d_{A,\emptyset}$ and d_B^1 for $d_{\emptyset,B}$.

Definition 3. *A* precubical set X *is a presheaf over* $\square$*, that is, a functor* $X : \square^{op} \to \mathbf{Set}$*. A* precubical map *between precubical sets is a natural transformation of functors.*

The value of X on the object U of $\square$ is denoted $X[U]$. For the face map associated to coface map $d_{A,B} : U \backslash (A \cup B) \to U$, we write $\delta_{A,B} = X[d_{A,B}] : X[U] \to X[U \backslash (A \cup B)]$. Elements of $X[U]$ are cells of X. For any $x \in X[U]$, elements of U are called events of x. We write $\mathsf{ev}(x) = U$. A precubical set is said to be finitely branching if every cell is the face of a finite number of cells.

Definition 4. *A* higher-dimensional automaton *(HDA)* $\mathcal{X}$ *is a triple* (X, i_X, F_X) *where* X *is a precubical set,* i_X *is a cell called the initial cell, and* F_X *is the set of final cells. An HDA map* $f : \mathcal{X} \to \mathcal{Y}$ *is a precubical map* $X \to Y$ *that preserves the initial cell*[4] *, that is,* $f(i_X) = i_Y$*. We denote* **HDA** *the category formed by HDAs as objects and HDA maps as morphisms.*

We assume that all HDAs are finitely branching.

Definition 5. *Let* S *be a conclist. The* standard S-cube *is the presheaf* $\square^S$ *represented by* S*, that is,*

- *for any* conclists T, $\square^S[T] = \mathbf{hom}_{\square}(T, S)$*;*
- $\square^S[(f, \varepsilon)](g, \eta) = (g, \eta) \circ (f, \varepsilon)$ *for* $(f, \varepsilon) \in \mathbf{hom}_{\square}(U, T)$*,* $(g, \eta) \in \square^S[T]$*.*

Example 6. Fig. 1 and Fig. 3 show examples of standard S-cubes, where $S = [a_1 \dashrightarrow a_2 \dashrightarrow a_3]$ in Fig 1 and $S = [a \dashrightarrow b]$ in Fig. 3.

Denote $\mathbf{y}_S \in \square^S[S]$ the cell that corresponds to the identity morphism. The following is a crucial result for this work and is implied by the Yoneda lemma.

Lemma 7. *Let* X *be a* precubical set*,* S *be a* conclist*, and* $x \in X$*. If* $\mathsf{ev}(x) = S$*, then there exists a unique precubical map* $\iota_x : \square^S \to X$ *such that* $\iota_x(\mathbf{y}_S) = x$*.*

[4] In our study, final cells are ignored because they are not relevant for bisimulation

3 Interval pomsets with interfaces vs. track objects

Pomsets are concurrent counterparts of words [30,25]. Interval pomsets [12] equipped with interfaces have been used to develop the language theory of HDAs [11,2,9,10]. They generalize the notion of conclist. Like standard cubes are presheaf representations of conclists, track objects are presheaf representations of interval pomsets with interfaces. As we proceed in this section, we revisit these concepts, then we present important new results for constructing the HDA-path category.

Background A *partially ordered multiset (pomset)* is a tuple $(P, <_P, \dashrightarrow_P, \lambda_P)$ where P is a finite set, $\lambda_P : P \to \Sigma$ is a labeling function over an alphabet Σ, $<_P$ is a strict partial order on P called *precedence order*, and $\dashrightarrow_P$ is a strict partial order on P called *event order* such that the relation $<_P \cup \dashrightarrow_P$ is total. Elements of P are called *events*. Intuitively, the latter condition means that any two events in P either are concurrent, thus can happen in parallel and ordered by $\dashrightarrow_P$, or occur sequentially, thus ordered by $<_P$. For $x, y \in P$, write $x \parallel y$ if x and y are *incomparable*, i.e, $x \neq y$, $x \not< y$, and $y \not< x$. We say that an element $x \in P$ is *minimal* if there is no element $y \in P$ such that $y < x$. Similarly, we say that x is *maximal* if there is no element $y \in P$ such that $x < y$. For $Q \subseteq P$, we say that Q is an *antichain* if $x \parallel y$ for all $x, y \in Q$. An antichain is maximal if it is not contained in another antichain. As the relation $<_P \cup \dashrightarrow_P$ is total, an antichain is a conclist.

A *partially ordered multiset with interfaces* or *ipomset* is a tuple $(P, <_P, \dashrightarrow_P, \lambda_P, S_P, T_P)$, where $(P, <_P, \dashrightarrow_P, \lambda_P)$ is a pomset, S_P is a subset of the $<$-minimal elements of P called *source interface*, and T_P is a subset of the $<$-maximal elements of P called *target interface*. The source and target interfaces are antichains, and thus, conclists. An ipomset P with empty precedence order, i.e $P = (U, \emptyset, \dashrightarrow_U, \lambda_U, S, T)$, is referred to as a *discrete ipomset* and will be denoted ${}_SU_T$. Pomsets may be regarded as ipomsets with empty interfaces. An *interval ipomset* is an ipomset P in which, for $x, y, z, w \in P$, if $x < z$ and $y < w$ then we have either $x < w$ or $y < z$.

Definition 8. *Let P and Q be ipomsets with $T_P \simeq S_Q$. The* gluing composition *of P and Q is $P * Q = ((P \sqcup Q)_{x \simeq f(x)}, <, \dashrightarrow, \lambda, S_P, T_Q)$, where $(P \sqcup Q)_{x \simeq f(x)}$ is the disjoint union of P and Q quotiented by the unique isomorphism $f : T_P \to S_Q$, and* $\lambda(x) = \begin{cases} \lambda_P(x) \text{ if } x \in P, \\ \lambda_Q(x) \text{ if } x \in Q, \end{cases}$ $\quad < \,=\, <_P \cup <_Q \cup (P \setminus T_P) \times (Q \setminus S_Q),$ $\quad \dashrightarrow \,=\, (\dashrightarrow_P \cup \dashrightarrow_Q)^+$

Definition 9. *Let P an ipomset. We define the order $\prec$ on maximal antichains of P as follows: $U \prec T$ iff $U \neq T$ and for all $u \in U$, $t \in T$, $t \not<_P u$.*

Proposition 10. *([9,18]) Let P be an ipomset. The following assertions are equivalent:*

1. *P is an interval order;*
2. *P is a finite gluing of discrete ipomsets;*
3. *The order $\prec$ defined on the maximal antichains of P is linear.*

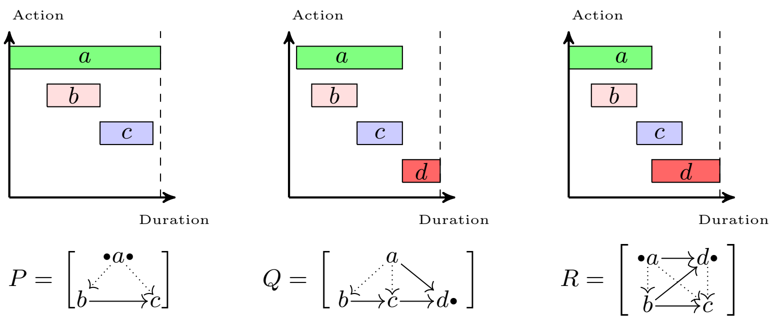


Fig. 5: Interval ipomsets (below) with their corresponding interval representations (above). An event with a dot on the left (resp. on the right) is an element of a source (resp. target) interface. Full arrows indicate precedence order, while dashed arrows indicates event order.

In this work, we focus solely on interval ipomsets: **all ipomsets are assumed to be interval even if not stated explicitly.**

Another instrumental tool for this work is track objects. They generalize the standard cubes of Def. 5, replacing a conclist with an interval ipomset.

Definition 11. *For a given interval ipomset P, the* track object *is an HDA* $(\square^P, i_{\square^P}, f_{\square^P})$ *where* $\square^P[U] = \hom_{\mathbf{IP}}(U, P)$, $\square^P[(f,\varepsilon)](g,\zeta) = (f,\varepsilon) \circ (g,\zeta)$, *and* $i_{\square^P} = (S_P \xrightarrow{\subseteq} P, \varepsilon)$ *and* $f_{\square^P} = (T_P \xrightarrow{\subseteq} P, \zeta)$ *where*

$$\varepsilon = \begin{cases} \lrcorner\!\!\ulcorner & \text{if } p \in S_P, \\ 0 & \text{if } p \notin S_P, \end{cases} \qquad \zeta = \begin{cases} \lrcorner\!\!\ulcorner & \text{if } p \in T_P, \\ 1 & \text{if } p \notin T_P. \end{cases}$$

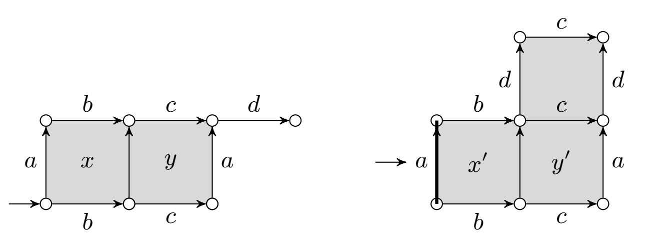


Fig. 6: Example of track objects: $\square^Q$ on the left and $\square^R$ on the right, where Q and R are the interval ipomsets of Fig. 5. The initial cell on the right is highlighted with a thick arrow.

Example 12. Fig. 6 shows examples of track objects. For instance, the cell x resp. y is modeled by $(f, x) \in \square^Q[U]$ resp. $(g, y) \in \square^Q[T]$, where $U = [b \dashrightarrow a]$, $T = [c \dashrightarrow a]$, and

It is easy to determine f and g as they are label preserving.

3.1 The category of interval ipomsets

The decomposition of an interval ipomset P into discrete ipomsets is not unique. However, there is a special decomposition that is unique with respect to specific properties, which we call the minimal discrete decomposition, defined as follows.

Definition 13. *The* minimal discrete decomposition *of an interval ipomset P into discrete ipomsets is $P = P_1 * \cdots * P_m$ where $P_i = (Q_i, \emptyset, \dashrightarrow_{|P_i}, \lambda_{|P_i}, S_i, T_i)$, Q_i are the maximal antichains*[5]*, $S_1 = S_P$, $T_m = T_P$, and $T_i = S_{i+1} = P_i \cap P_{i+1}$.*

The minimal discrete decomposition plays a central role in our work. We rely heavily on it throughout the forthcoming proofs. Notably, such decomposition is unique.

Example 14. The minimal discrete decompositions of interval ipomsets of Fig. 5 are as follows:

$$P = \begin{bmatrix} \bullet a \bullet \\ b \end{bmatrix} * \begin{bmatrix} \bullet a \bullet \\ c \end{bmatrix}; \qquad Q = \begin{bmatrix} a \bullet \\ b \end{bmatrix} * \begin{bmatrix} \bullet a \\ c \end{bmatrix} * d\bullet; \qquad R = \begin{bmatrix} \bullet a \bullet \\ b \end{bmatrix} * \begin{bmatrix} \bullet a \\ c \bullet \end{bmatrix} * \begin{bmatrix} d \bullet \\ \bullet c \end{bmatrix}.$$

Since the notion of interval ipomset generalizes the concept of conclist, it is convenient to think about a category with interval ipomsets as objects that extends the $\square$ category.

Definition 15. *The category* **IP** *consists of the following.*

- *Objects are interval ipomsets;*
- *A morphism*[6] *between two interval ipomsets P and Q is a pair (f, ε) such that $f : P \to Q$ is an injective map that reflects the precedence order, that is, for $x, y \in P$, if $f(x) <_Q f(y)$ then $x <_P y$ and preserves the essential event order, i.e, for $x \parallel y \in P$ if $x \dashrightarrow_P y$ then $f(x) \dashrightarrow_Q f(y)$; and $\varepsilon : Q \to \{0, \lrcorner\!\ulcorner, 1\}$ such that $f(P) = \varepsilon^{-1}(\lrcorner\!\ulcorner)$ and if $q <_Q q'$ then $(\varepsilon(q), \varepsilon(q')) \in \preceq_{ipom}$, where $\preceq_{ipom} = \{(1,1), (0,0), (\lrcorner\!\ulcorner, \lrcorner\!\ulcorner), (1, \lrcorner\!\ulcorner), (1,0), (\lrcorner\!\ulcorner, 0)\}$*
- *The composition of morphisms $(f, \varepsilon) : P \to Q$ and $(g, \zeta) : Q \to R$ is $\big((g,\zeta) \circ (f,\varepsilon)\big) = (g \circ f, \eta)$, where $\eta(u) = \begin{cases} \varepsilon(g^{-1}(u)) & \text{for } u \in g(Q), \\ \zeta(u) & \text{otherwise.} \end{cases}$*

We write $P \simeq Q$ if there exists a bijective map $f : P \to Q$ such that f is also an order isomorphism. If such an isomorphism exists, then it is unique [9].

The definition of the morphisms of the category **IP** will serve later to define the track objects (Def 11). The intuition for the values of $\varepsilon(q)$ is to be 1 if the event q happens before the events of $f(P)$, $\lrcorner\!\ulcorner$ if the event q is in $f(P)$, and 0 if the event q happens after the events of $f(P)$. That is why we allow all possible cases for $(\varepsilon(q), \varepsilon(q'))$, in $\preceq_{ipom}$, except the cases where q' terminate while $q \in f(P)$ so we eliminate pairs $(\lrcorner\!\ulcorner, 1)$ and the case where q has not started yet while $q' \in f(P)$ so we eliminate $(0, \lrcorner\!\ulcorner)$.

[5] In this case, we have $Q_1 \prec \cdots \prec Q_m$

[6] For morphisms of **IP**, we do not care about interfaces

Definition 16. *Let P and R be composable ipomsets. There are two morphisms related to the gluing*[7] $P * R$:
- initial inclusions $i_P^{P*R} = (P \subseteq P * R, \varepsilon)$, *and*
- final inclusions $f_R^{P*R} = (R \subseteq P * R, \zeta)$, *where*

$$\varepsilon(x) = \begin{cases} \lrcorner\!\ulcorner & \textit{for } x \in P, \\ 0 & \textit{otherwise}, \end{cases} \qquad \zeta(x) = \begin{cases} \lrcorner\!\ulcorner & \textit{for } x \in R, \\ 1 & \textit{otherwise}. \end{cases}$$

Definition 17. *For a conclist S, let $\mathbf{IP}_S^0 \subseteq \mathbf{IP}$ be a subcategory with ipomsets P with $S_P = S$ as objects and morphisms*[8] $\hom_{\mathbf{IP}_S^0}(P, Q) = \{i_P^{P*R} \mid R$ *is an ipomset such that* $P * R \cong Q\}$. *We define the category* $\mathbf{IP}^0 = \bigcup_{S \in \square} \mathbf{IP}_S^0$.

Example 18. The following are initial inclusions.
1. $R_1 : \begin{bmatrix} \bullet a \bullet \\ b \end{bmatrix} \to \begin{bmatrix} \bullet a \bullet \\ b \longrightarrow c \end{bmatrix}$ where $R_1 = \begin{bmatrix} \bullet a \bullet \\ c \end{bmatrix}$;
2. $R_2 : \begin{bmatrix} \bullet a \\ b \longrightarrow c \end{bmatrix} \to \begin{bmatrix} \bullet a \longrightarrow d \bullet \\ b \longrightarrow c \end{bmatrix}$, where $R_2 = \begin{bmatrix} d \bullet \\ \bullet c \end{bmatrix}$.

We use categories $\mathbf{IP}_S^0$ to construct the bisimulation and the modal logic later in Section 5. In the next section, we show how we may regard $\mathbf{IP}^0$ as a subcategory of **HDA**, as required to apply the open map technique.

3.2 Defining the HDA-path category

Theorem 19. *The functor* $\mathsf{Tr} : \mathbf{IP} \to \mathbf{HDA}$, *given by formulas* $\mathsf{Tr}(P) = \square^P$ *and* $\mathsf{Tr}(f, \varepsilon)(g, \zeta) = (f, \varepsilon) \circ (g, \zeta)$ *for* $(f, \varepsilon) \in \hom_{\mathbf{IP}}(P, Q)$, *is faithful.*

Definition 20. *The category* $\mathbf{T}^0$ *is the subcategory of* **HDA** *given by* $\mathbf{T}^0 = \mathsf{Tr}(\mathbf{IP}^0)$. *Thus, it is defined by track objects as objects and morphisms are* $\mathbf{i}_P^{P*R} = \mathsf{Tr}(i_P^{P*R})$, *for* $\mathbf{i}_P^{P*R}$ *defined in Def. 16, and called* initial inclusions.

Similarly, we call $\mathsf{Tr}(f_P^{P*R})$, for f_P^{P*R} defined in Def. 16, a *final inclusion,* and we write $\mathbf{f}_P^{P*R}$.

Definition 21. *Let $\square^Q$ and $\square^R$ be track objects such that $T_Q \simeq S_R \simeq U$. The* gluing composition *of $\square^Q$ and $\square^R$ is the pushout HDA* $(\square^Q * \square^R, I_{\square^Q}, F_{\square^R})$ *where* $\square^Q * \square^R = \operatorname{colim}(\square^R \xleftarrow{\mathbf{i}_U^R} \square^U \xrightarrow{\mathbf{f}_U^Q} \square^Q)$

Lemma 22. *([9]) If Q and R are composable ipomsets, then* $\square^{Q*R} = \square^Q * \square^R$. *In addition,* $\mathbf{i}_Q^{Q*R}(g, \zeta) = (g, \mathbf{i}_Q^{Q*R}(\zeta))$ *and* $\mathbf{f}_R^{Q*R}(g, \zeta) = (g, \mathbf{f}_R^{Q*R}(\zeta))$ *are given by*

$$\mathbf{i}_Q^{Q*R}(\zeta)(p) = \begin{cases} \zeta(p) & \textit{for } p \in Q, \\ 0 & \textit{otherwise}, \end{cases} \qquad \mathbf{f}_R^{Q*R}(\zeta)(p) = \begin{cases} \zeta(p) & \textit{for } p \in R, \\ 1 & \textit{otherwise}. \end{cases}$$

Proposition 23. *Let $\square^P$ be a track object. If $P = P_1 * P_2 * \cdots * P_m$ is the minimal discrete decomposition of P, then* $\square^P = \square^{P_1} * \cdots * \square^{P_m}$.

Note that $\hom_{\mathbf{T}^0}(\square^P, \square^Q) \cong \{\square^R \mid \square^Q = \square^P * \square^R\}$ by Lem. 22 and Prop. 23.

[7] We regard both P and Q as sub-pomsets of $P * Q$.

[8] Since i_P^{P*R} is uniquely determined by R, we might identify i_P^{P*R} and R.

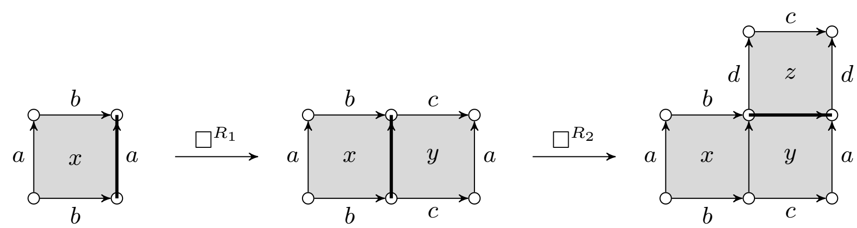


Fig. 7: Examples of morphisms of $\mathbf{T}^0$. R_1 and R_2 are initial inclusions of Ex. 18

4 Paths and tracks

The computations or runs of HDAs, which track traversed cells and face maps, have been modeled by paths in [14] and by tracks in the categorical framework [9]. In the following subsection, we revisit these concepts and then in Subsection 4.2 establish a relation between them. This link is crucial for expressing the satisfaction relation of IPML on paths, similarly to temporal logic.

4.1 Background: Paths and their labels

Definition 24. *A* path *in a precubical set X is a sequence $\alpha = (x_0, \varphi_1, x_1, \varphi_2, \dots, \varphi_n, x_n)$, where $x_k \in X[U_k]$ are cells, and for all k, either*
- *$\varphi_k = d^0_A \in \square(U_{k-1}, U_k)$, $A \subseteq U_k$ and $x_{k-1} = \delta^0_A(x_k)$ (up-step), or*
- *$\varphi_k = d^1_B \in \square(U_k, U_{k-1})$, $B \subseteq U_{k-1}$, $\delta^1_B(x_{k-1}) = x_k$ (down-step).*

We write $x_{k-1} \nearrow^A x_k$ for the up-steps and $x_{k-1} \searrow_A x_k$ for the down-steps in α. Intuitively, moving by an up step $x_{k-1} \nearrow^A x_k$ means that the list of events A started and became active in the next cell x_k. Similarly, moving by a down step $x_{k-1} \searrow_A x_k$ means that the list of events A terminated and became inactive in the cell x_k. For a path written as above, we write $\mathsf{start}(\alpha)$ and $\mathsf{end}(\alpha)$ for the first cell x_0 and the final cell x_m, respectively. We write Path_X for the set of all paths on a precubical set X.

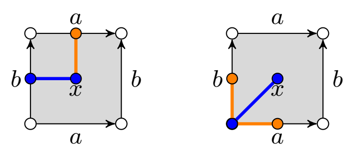
A precubical map $f : X \to Y$ induces a map $f : \mathrm{Path}_X \to \mathrm{Path}_Y$. For α denoted as above, $f(\alpha)$ is the path $\big(f(x_0), \varphi_1, f(x_1), \varphi_2, \dots, \varphi_n, f(x_n)\big)$. We say that a path is *sparse* if its steps are alternating between up-steps and down-steps. The *concatenation* of α denoted as above and $\beta = (y_0, \psi_1, y_1, \psi_2, \dots, \psi_m, y_m)$, defined if $x_n = y_0$, is the path $\alpha * \beta$ given by $\alpha * \beta = (x_0, \varphi_1, x_1, \varphi_2, \dots, \varphi_n, x_n, \psi_1, y_1, \psi_2, \dots, \psi_m, y_m)$.

Definition 25. *The label of a path α is the ipomset $\mathsf{ev}(\alpha)$, computed recursively:*
1. *If $\alpha = (x)$ has length 0, then $\mathsf{ev}(\alpha) = {}_{\mathsf{ev}(x)}\mathsf{ev}(x)_{\mathsf{ev}(x)}$.*
2. *If $\alpha = \big(y \nearrow^A x\big)$, where $A \subseteq \mathsf{ev}(x)$, then $\mathsf{ev}(\alpha) = {}_{\mathsf{ev}(x)\setminus A}\mathsf{ev}(x)_{\mathsf{ev}(x)}$.*
3. *If $\alpha = (x \searrow_B y)$, where $B \subseteq \mathsf{ev}(x)$, then $\mathsf{ev}(\alpha) = {}_{\mathsf{ev}(x)}\mathsf{ev}(x)_{\mathsf{ev}(x)\setminus B}$.*
4. *If $\alpha = \beta_1 * \dots * \beta_n$, where β_i are steps, then $\mathsf{ev}(\alpha) = \mathsf{ev}(\beta_1) * \dots * \mathsf{ev}(\beta_n)$.*

As a finite gluing of discrete ipomsets, by Prop. 10, $\mathsf{ev}(\alpha)$ is an interval ipomset.

Definition 26. *Let* $\alpha = (x_0, \varphi_1, x_1, \ldots, \varphi_n, x_n)$*. We say that* β *is a* restriction *of* α *and write* $\beta \overset{0}{\hookrightarrow} \alpha$*, if* $\beta = \left(x_0, \varphi_1, x_1, \varphi_2, \ldots, x_{j-1}, \varphi'_j, x'_j\right)$*, where* $j \leq n$ *and*
- *if* $\varphi_j = d^1_B$ *then* $\varphi'_j = d^1_A$ *for* $A \subseteq B$*;*
- *if* $\varphi_j = d^0_B$ *then* $\varphi'_j = d^0_A$ *for* $A \subseteq B$ *and* $x'_j = \delta^0_{B \setminus A}(x_j)$*.*

Example 27. On the left, the path $(\delta^0_a x \nearrow^a x)$ in blue is a restriction of the path $(\delta^0_a x \nearrow^a x \searrow_b \delta^1_b x)$ in orange. On the right, the paths $(\delta^0_a \delta^0_b x \nearrow^a \delta^0_b x)$ and $(\delta^0_a \delta^0_b x \nearrow^b \delta^0_b x)$ in orange are restrictions of $\alpha_2 = (\delta^0_a x \nearrow^a x)$ in blue.



Definition 28. Congruence *of paths is the equivalence relation generated by* $(x \nearrow^A y \nearrow^B z) \simeq x \nearrow^{A \cup B} z$*,* $(x \searrow^A y \searrow^B z) \simeq x \searrow^{A \cup B} z$*, and if* $\alpha \simeq \alpha'$ *then* $\gamma * \alpha * \beta \simeq \gamma * \alpha' * \beta$*.*

If $\alpha \simeq \beta$*, then* $\mathsf{start}(\alpha) = \mathsf{start}(\beta)$ *and* $\mathsf{end}(\alpha) = \mathsf{end}(\beta)$*. Furthermore, every path* α *is congruent to a unique sparse path, which is denoted* $sp(\alpha)$*.*

Definition 29. *A* track *in a precubical set* X *is a precubical map* $g : \square^P \to X$ *where* P *is an ipomset.*

In the case of a track in an HDA (X, i_X, I_F), we say that g is an initial track if P is a discrete ipomset and $g(\mathbf{y}_P) = i_X$.

4.2 The categories of tracks and paths

The relation $\simeq$ is an equivalence relation, which allows the following definition.

Definition 30. *Let* X *be a precubical set. We define the category* $\mathbb{P}_X$ *as follows.*
- *Objects are equivalence classes of paths with respect to* $\simeq$*.*
- *Morphisms are* $\hom_{\mathbb{P}_X}(P[\alpha], [\beta]) = \{[\gamma] \mid \alpha * \gamma \simeq \beta\}$*), called* path extensions *and denoted* $e^{\alpha * \gamma}_{\alpha} = [\alpha] \to [\alpha * \gamma]$*.*
- *The composition of* $e^{\alpha*\beta}_{\alpha}$ *and* $e^{\alpha*\beta*\gamma}_{\alpha*\beta}$ *is* $e^{\alpha*\beta*\gamma}_{\alpha}$*.*

Let $p : \square^P \to X$ and $q : \square^Q \to X$ be two tracks such that $p(F_P) = q(I_Q)$. By Lem. 22, p and q glue to a track $p * q : \square^{P*Q} \to X$, called the *the gluing of tracks* p and q, that satisfies $(p * q) \circ \mathbf{i}^{P*Q}_P = p$ and $(p * q) \circ \mathbf{f}^{P*Q}_Q = q$.

Definition 31. *Let* X *be a precubical set. We define the category of tracks* $\mathbb{T}_X$ *as follows.*
- *Objects are tracks* $p : \square^P \to X$*;*
- *Morphisms are* $\hom_{\mathbb{T}_X}(p, q) = \{r \mid q = p * r\}$*), called* track extensions *and denoted* $\mathbf{e}^{P*R}_P$*, where* $p : \square^P \to X$*,* $r : \square^R \to X$ *and thus* $q : \square^{P*R} \to X$*. In other words, there is a morphism* $\mathbf{e}^{P*R}_P$ *between tracks* $p : \square^P \to X$ *and*

$p' : \square^Q \to X$ iff $Q = P * R$ and they are related by the left triangle in the following diagram:

$$\begin{array}{ccccc} \square^P & \xrightarrow{\mathbf{i}_P^{P*R}} & \square^{P*R} & \xrightarrow{\mathbf{i}_{P*R}^{P*R*Q}} & \square^{P*R*Q} \\ & \searrow^{p} & \downarrow{p'} & \swarrow^{p''} & \\ & & X & & \end{array}$$

– *Composition of $\mathbf{e}_P^{P*R} : p \to p'$ and $\mathbf{e}_{P*R}^{P*R*Q} : p' \to p''$ is $\mathbf{e}_P^{P*R*Q} : p \to p''$, as shown in the diagram above.*

The Yoneda lemma 7 is based on the unique cell $\mathbf{y}_S$ of a conclist S. For an ipomset P, we introduce the characteristic path ρ_P that allows a generalization of the Yoneda lemma by substituting ρ_P for $\mathbf{y}_S$. It is a key contribution of this work that will be used to bridges tracks and paths.

Definition 32. *Consider an interval ipomset P and $P = P_1 * P_2 * \cdots * P_m$ its minimal discrete decomposition with $P_i = {}_{U\setminus A}U_{U\setminus B}$. The characteristic path of P is $\rho_P = \beta_1 * \beta_2 \cdots * \beta_m$ the concatenation of steps $\beta_i = (\delta_A^0(\mathbf{y}_U), \mathbf{y}_U, \delta_B^1(\mathbf{y}_U))$.*

Note that the characteristic path is the sparse path $\rho_P \in P_{\square^P}$ such that $\mathsf{ev}(\rho) = P$, $\mathsf{start}(\rho) = I_{\square^P}$, and $\mathsf{end}(\rho) = F_{\square^P}$, calculated by induction as follows.

– *If $P = {}_{U\setminus A}U_{U\setminus B}$ is discrete, then $\rho_P = (\delta_A^0(\mathbf{y}_U), \mathbf{y}_U, \delta_B^1(\mathbf{y}_U))$.*
– *If $P = R * Q$, then $\rho_P = \mathbf{i}_R^{R*Q}(\rho_R) * \mathbf{f}_Q^{R*Q}(\rho_Q)$[9], where $\rho_R \in P_{\square^R}$ and $\rho_Q \in P_{\square^Q}$ the characteristic paths of R and Q respectively.*

The following Prop. generalizes Yoneda Lemma 7. Instead of cells, here we have paths, and instead of conclist, we have ipomsets.

Proposition 33. *Let X be a precubical set, P an ipomset, $\alpha \in \mathrm{Path}_X$. If $\mathsf{ev}(\alpha) = P$, then there exist a unique $\rho'_P \simeq \rho_P$ and a unique track $g_\alpha : \square^P \to X$ such that $g_\alpha(\rho'_P) = \alpha$.*

The track g_α depends on the class of α up to $\simeq$ rather than α:

Lemma 34. *If $\alpha \simeq \beta$, then $g_\alpha = g_\beta$.*

Theorem 35. *For any precubical set X, the categories $\mathbb{P}_X$ and $\mathbb{T}_X$ are isomorphic.*

5 Bisimulation and modal logic

Fix a conclist S and denote by $\mathbf{HDA}_S$ the full subcategory of $\mathbf{HDA}$ with HDAs (X, i_X, F_X) such that $\mathsf{ev}(i_X) = S$. So that we have $\mathbf{HDA} = \bigcup_{S\in\square} \mathbf{HDA}_S$. Similarly, $\mathbf{T}_S^0$ is the category that has track objects $\square^P$, with $S_P \simeq S$, as objects and initial inclusions as morphisms. In this section, we apply the open map bisimulation technique [19] with $\mathbf{T}_S^0$ as the HDA-path category to define the $\mathbf{T}^0$-*bisimulation* and then to define the IPML.

[9] We can check that $\mathbf{i}_R^{R*Q}(\rho_R)$ and $\mathbf{f}_Q^{R*Q}(\rho_Q)$ can be concatenated by elementary calculations, using the expression of initial and final inclusion of Lem. 22 and of the initial and final cells in Def. 11.

Overview A morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{HDA}_S$ has the *path lifting-property* with respect to $\mathbf{T}^0_S$ if whenever for $\mathbf{i}^Q_P \in \hom_{\mathbf{T}^0_S}$ (thus $Q \cong P * R$ for some ipomset R), $p : \square^P \to X$ and $q : \square^Q \to Y$, $q \circ \mathbf{i}^{P*R}_P = \varphi \circ p$ i.e the following diagram commutes,

$$\begin{array}{ccc} \square^P & \xrightarrow{\;p\;} & X \\ {\scriptstyle \mathbf{i}^{P*R}_P}\big\downarrow & \overset{p'}{\nearrow} & \big\downarrow{\scriptstyle\varphi} \\ \square^{P*R} & \xrightarrow[\;q\;]{} & Y \end{array}$$

then there exists a track p' such that $p' \circ \mathbf{i}^{P*R}_P = p$ and $\varphi \circ p' = q$ i.e the two triangles in the previous diagram commute. In this case, we say that φ is $\mathbf{T}^0_S$*-open* or that φ is *open with respect to* $\mathbf{T}^0_S$. This gives rise to a notion of bisimulation with respect to $\mathbf{T}^0_S$.

5.1 Bisimulation from open maps for HDA

Definition 36. *Let $\mathcal{Y}$, $\mathcal{Z}$ be HDAs. We say that $\mathcal{Y}$ and $\mathcal{Z}$ are $\mathbf{T}^0_S$-bisimilar if there is a span of $\mathbf{T}^0_S$-open HDA maps $\mathcal{Y} \xleftarrow{\varphi} \mathcal{X} \xrightarrow{\psi} \mathcal{Z}$ with a common HDA $\mathcal{X}$.*

A path α in an HDA $\mathcal{X}$ is a path in the precubical set X such that $\mathsf{start}(\alpha) = i_X$. We denote $\mathrm{Path}_{\mathcal{X}}$ the set of paths in $\mathcal{X}$. Similarly, we denote $\mathbb{P}_{\mathcal{X}}$ the category of classes of paths in an HDA. A morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ in **HDA** has the *future path lifting property* if for $\alpha \in \mathrm{Path}_{\mathcal{X}}$ and $\beta \in \mathrm{Path}_Y$, if $\varphi(\alpha)$ and β can be concatenated, then there exists α' in X such that α and α' can be concatenated and $\varphi(\alpha * \alpha') = \varphi(\alpha) * \beta$.

Lemma 37. *For any HDA map $\varphi : \mathcal{X} \to \mathcal{Y}$, φ is $\mathbf{T}^0_S$-open iff φ has the future path lifting.*

Definition 38. *A* closed cell-bisimulation *between HDAs $\mathcal{Y}$ and $\mathcal{Z}$ is a relation $\overline{R}$ between cells in Y and Z such that*

1. *initial cells i_Y and i_Z are related;*
2. *$\overline{R}$ respects labels: for all $(y,z) \in \overline{R}$, $\mathsf{ev}_Y(y) = \mathsf{ev}_Z(z)$;*
3. *if $(y,z) \in \overline{R}$, then $(\delta^\nu_A(y), \delta^\nu_A(z)) \in \overline{R}$ for $A \subseteq \mathsf{ev}_Y(y) = \mathsf{ev}_Z(z)$, $\nu \in \{0,1\}$;*
4. *for all $(y,z) \in \overline{R}$, if there exists y' such that $\delta^0_A(y') = y$ for some $A \subseteq \mathsf{ev}(y')$, then there exists z' such that $\delta^0_A(z') = z$ and $(y', z') \in \overline{R}$;*
5. *for all $(y,z) \in \overline{R}$, if there exists z' such that $\delta^0_A(z') = z$ for some $A \subseteq \mathsf{ev}(z')$, then there exists y' such that $\delta^0_A(y') = y$ and $(y', z') \in \overline{R}$;*

A cell x in an HDA $\mathcal{X}$ is said to be accessible if there exists $\alpha_x \in \mathrm{Path}_{\mathcal{X}}$ such that $\mathsf{end}(\alpha_x) = x$, we denote $\mathcal{X}_{acc}$ the set of accessible cells in $\mathcal{X}$.

Definition 39. *A* cell-bisimulation *between $\mathcal{Y}$ and $\mathcal{Z}$ is a relation R between $\mathcal{Y}_{acc}$ and $\mathcal{Z}_{acc}$ that satisfies the same properties as Def. 38, replacing 3. by*

3. *for all $(y,z) \in R$, for all $A \subseteq \mathsf{ev}_Y(y) = \mathsf{ev}_Z(z)$,*
   (a) *$(\delta^1_A(y), \delta^1_A(z)) \in R$*
   (b) *$\delta^0_A(y) \in \mathcal{Y}_{acc}$ iff , $\delta^0_A(z) \in \mathcal{Z}_{acc}$. In this case, $(\delta^0_A(y), \delta^0_A(z)) \in R$*

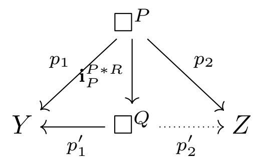
Theorem 40. *Two HDAs $\mathcal{Y}$ and $\mathcal{Z}$ are $\mathbf{T}^0_S$-bisimilar iff they are cell-bisimilar.*

5.2 Modal characterization

In this section, we delve into the core contributions of our work. Initially, we introduce the concept of track bisimulation, followed by a formal presentation of the Ipomset modal logic. Notably, the notion of track bisimulation serves as a crucial link connecting our logic's modalities with the existing concept of ST-bisimulation found in the literature. More specifically, it will demonstrate that our logic characterizes the notion of ST-bisimulation.

Definition 41. *A* track-bisimulation*, with respect to* $\mathbf{T}_S^0$*, between HDAs* $\mathcal{Y}$ *and* $\mathcal{Z}$ *is a symmetric relation* R *of pairs of tracks* (p_1, p_2) *with common domain* $\square^P$*, so* $p_1 : \square^P \to Y$ *is a track in* Y *and* $p_2 : \square^P \to Z$ *is a track in* Z*, such that*

1. *initial tracks* $\iota_{\mathcal{X}} : \square^S \to X$ *and* $\iota_{\mathcal{Y}} : \square^S \to Y$ *are related;*
2. *For* $(p_1, p_2) \in \mathsf{R}$*, if* $p'_1 \circ \mathbf{i}_P^{P*R} = p_1$*, in the diagram*



then there is p'_2 *such that* $(p'_1, p'_2) \in \mathsf{R}$ *and* $p'_2 \circ \mathbf{i}_P^{P*R} = p_2$*.*

We say that a track-bisimulation is strong *if, in addition, it satisfies:*

3. *If* $(p_1, p_2) \in \mathsf{R}$ *for* $p_1 : \square^Q \to Y$*,* $p_2 : \square^Q \to Z$*, then for every* $\mathbf{i}_P^{P*R} : \square^P \to \square^Q \in \mathbf{T}_S^0$ *we have* $(p_1 \circ \mathbf{i}_P^{P*R}, p_2 \circ \mathbf{i}_P^{P*R}) \in \mathsf{R}$*.*

We say that two HDAs are (strong) track-bisimilar *iff there is a* (strong) track-bisimulation *between them.*

We introduce the novel modal logic IPML with HDAs as models, following the approach of Nielsen and Winskel [19].

Definition 42 (Ipomset Modal Logic). *The set of formulae in Ipomset Modal Logic (IPML) is given by the following syntax:*

$$F, G ::= \top \mid \bot \mid F \wedge G \mid F \vee G \mid \langle \mathbf{i}_P^{P*R} \rangle F \mid \overline{\langle \mathbf{i}_P^{P*R} \rangle} F,$$

where $\mathbf{i}_P^{P*R}$ *is a morphism in* $\mathbf{T}_S^0$*. The modality* $\overline{\langle \mathbf{i}_P^{P*R} \rangle}$ *is a backward modality, while* $\langle \mathbf{i}_P^{P*R} \rangle$ *is a forward modality.*

Like Nielsen and Winskel's original approach, IPML should also have infinitary conjunctions. In contrast, we only consider HDAs with finitely branching, for which no infinitary conjunction is required.

The satisfaction relation between a track $p : \square^P \to X$ and a formula F is given by structural induction on assertions as follows:

- $p \models \top$ for all p, $p \models \bot$ for no p, $p \models F \wedge G$ iff $p \models F$ and $p \models G$, and $p \models F \vee G$ iff $p \models F$ or $p \models G$;
- $p \models \langle \mathbf{i}_P^{P*R} \rangle F$ where $\mathbf{i}_P^{P*R} : \square^P \to \square^{P*R}$, iff there exists is a track $q : \square^{P*R} \to X$ for which $q \models F$ and $p = q \circ \mathbf{i}_P^{P*R}$;
- $p \models \overline{\langle \mathbf{i}_Q^P \rangle} F$ where $\mathbf{i}_Q^P : \square^Q \to \square^P$, with $P = Q * S$, iff there exists a track $q : \square^Q \to X$ for which $q \models F$ and $q = p \circ \mathbf{i}_Q^{Q*S}$.

By Th. 35, the previous modal logic, given with satisfaction relation on tracks, induces a modal logic interpreted over paths, where congruent paths satisfy the same formulas. The induced satisfaction relation is thus a binary relation $\models$ that relates $\alpha \in \mathbb{P}_{\mathcal{X}}$, with $\mathsf{ev}(\alpha) = P$, to formulae.

For a given track, the forward modality is uniquely determined by the choice of the extending ipomset R. While the backward modality is uniquely determined by the decomposition of P into two ipomsets. Thus, our modalities could be reformulated, as follows.

- $\alpha \models \langle R \rangle F$ with R an ipomset iff there is $\beta \in \mathbb{P}_X$ for which $\alpha * \beta \models F$ and $\mathsf{ev}(\beta) = R$;
- $\alpha \models \overline{\langle Q * S \rangle} F$ with $P = Q * S$ iff there is $\alpha' \overset{0}{\hookrightarrow} \alpha$ in $\mathbb{P}_{\mathcal{X}}$ for which $\mathsf{ev}(\alpha') = Q$ and $\alpha' \models F$.

Example 43. Consider the paths in the HDAs of Fig.8. We have the following:

- $(i_{X_2}) \models \big\langle \big[\begin{smallmatrix} c\bullet \\ \bullet a \bullet \end{smallmatrix} \big] \big\rangle \big\langle [\bullet a \to d \bullet] \big\rangle \top$, meaning that there is a path α_2 labeled by $\big[\begin{smallmatrix} c\bullet \\ \bullet a \bullet \end{smallmatrix} \big]$ from which it is possible to terminate an event labeled by a and start an event labeled by d, by executing the path β_2.
- $\alpha_1 \models \overline{\big\langle c * \big[\begin{smallmatrix} d \\ a \end{smallmatrix} \big] \big\rangle} \big\langle \big[\begin{smallmatrix} b \\ d \end{smallmatrix} \big] \big\rangle \top$, meaning that there exists a restriction α_1' of α_1 such that $\mathsf{ev}(\alpha_1') = c$ and $\alpha_1' \models \big\langle \big[\begin{smallmatrix} b \\ d \end{smallmatrix} \big] \big\rangle \top$, that is, α_1' can be concatenated with a path β_1 labeled by $\big[\begin{smallmatrix} b \\ d \end{smallmatrix} \big]$.

[19, Thm. 15] now immediately implies the following.

Theorem 44. *HDAs are (strong) track-bisimilar iff initial tracks satisfy the same forward (and backward) assertions.*

Theorem 45. *If HDAs are $\mathbf{T}^0_S$-bisimilar, then they are strong track-bisimilar.*

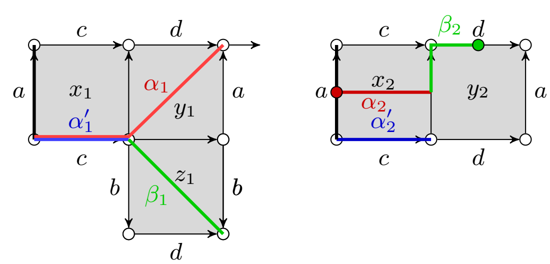


Fig. 8: Two HDAs $\mathcal{X}_1$ and $\mathcal{X}_2$ in $\mathbf{HDA}_a$ that are strong track-bisimilar, cell-bisimilar, but not closed cell-bisimilar HDAs. Each HDA has the edge labeled by a with a thick line as the unique initial cell.

It is clear that closed cell-bisimilarity implies cell-bisimilarity. The following example shows that the opposite direction is false. It also shows that track-bisimilarity does not imply closed cell bisimilarity. The opposite direction of the later remains an open question that we would like to answer in an extended version of this work.

Example 46. Fig. 8 shows two HDAs $\big(X_1, \delta_c^0(x_1)\big)$ (on the left) and $\big(X_2, \delta_c^0(x_2)\big)$ (on the right) that are track-bisimilar but not closed cell-bisimular. Let $P_1 = \left[\begin{smallmatrix} \bullet a \bullet \\ c \end{smallmatrix}\right]$, $P_2 = \left[\begin{smallmatrix} \bullet a \\ d \bullet \end{smallmatrix}\right]$, $P_3 = \left[\begin{smallmatrix} \bullet d \\ b \end{smallmatrix}\right]$. It is not difficult to check that $K = \{(g, g') \mid g : \square^P \to X_1, g' : \square^P \to X_2 \mid \text{there exists } i_P^{P_1 * P_2}\}$ is a strong track-bisimulation. However, they cannot be closed cell-bisimilar, because if there is a closed cell-bisimulation between them, then $\delta_a^0(y_1)$ and $\delta_a^0(y_2)$ are related. However, $\delta_b^0(z_1) = \delta_a^0(y_1)$ while there exists no cell $z_2 \in X_2$ such that $\delta_b^0(z_2) = \delta_a^0(y_2)$.

Remark 47. To check the track bisimilarity of the HDAs of the previous example, one may check that initial *paths* satisfy the same forward assertions. Note that if we allow i_{X_1} and i_{X_2} to be the nodes $\delta_{ac}^0(x_1)$ and $\delta_{ac}^0(x_2)$ respectively, $\mathcal{X}_1$ and $\mathcal{X}_2$ will no longer be track bisimilar. Due to the distinguishing formulae $\big\langle(c)\big\rangle\big\langle\left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right]\big\rangle\top$ that holds in (i_{X_1}) but not in (i_{X_2}). In fact, in this case, we will have equivalence between the notions of strong track bisimilarity and cell-bisimilarity.

We say that paths $\alpha = (x_0, \varphi_1, x_1, \varphi_2, \ldots, \varphi_n, x_n)$ and $\beta = (y_0, \psi_1, y_1, \psi_2, \ldots, \psi_m, y_m)$ have the same shape if $n = m$ and $\varphi_i = \psi_i$ for all i. The following notion of behavioral equivalence was originally introduced by van Glabbeek [14] as ST-bisimulation. In our setting it has been formulated in [9] as follows.

Definition 48. *A* path-bisimulation *between HDAs $\mathcal{Y}$ and $\mathcal{Z}$ is a symmetric relation R between paths in Y and Z such that*

1. *initial paths (i_Y) and (i_Z) are related;*
2. *R respects the shape: for all $(\rho, \sigma) \in \mathsf{R}$, ρ and σ have the same shape;*
3. *for all $(\rho, \sigma) \in \mathsf{R}$ and path ρ' in Y where ρ and ρ' may be concatenated, there exists a path σ' in Z such that $(\rho * \rho', \sigma * \sigma') \in \mathsf{R}$;*

A path-bisimulation is called strong *if, in addition, it satisfies:*

4. *for all $(\rho, \sigma) \in \mathsf{R}$ and ρ' a restriction of ρ, there exists σ' a restriction of σ such that $(\rho', \sigma') \in \mathsf{R}$.*

Finally, $\mathcal{X}$ and $\mathcal{Y}$ are (strong) path-bisimilar if there exists a (strong) path-bisimulation R *between them; this is an equivalence relation.*

Theorem 49. *Two HDAs $\mathcal{X}_1$ and $\mathcal{X}_2$ are (strong) track-bisimilar iff they are (strong) path-bisimilar.*

Conclusion We have investigated open maps for the category $\mathbf{T}^0 \subseteq \mathbf{T}$. The general approach yields the abstract notion of $\mathbf{T}^0$-bisimulation and a path logic, IPML (with past modality) for which the logical equivalence is equivalent to the (strong) Track-bisimulation. We showed that our logic is powerful enough to capture true concurrency and characterize (strong) Path-bisimulation, known in the literature as ST-bisimulation. We summarize the hierarchy of the different notions in Fig.4. In future work, we aim to look at the extension of IPML that captures the finest bisimulation equivalence, hereditary history preserving bisimulation. We would thus have a complete spectrum for concurrency bisimulation notions that might be interpreted over other models of concurrency such as Petri nets, event structures, and configuration structures, due to the expressiveness of HDA.

References

1. Aceto, L., Ingolfsdottir, A., Larsen, K., Srba, J.: Reactive Systems: Modelling, Specification and Verification. Cambridge University Press (2007)
2. Amrane, A., Bazille, H., Fahrenberg, U., Ziemiański, K.: Closure and decision properties for higher-dimensional automata. In: Ábrahám, E., Dubslaff, C., Tarifa, S.L.T. (eds.) Theoretical Aspects of Computing – ICTAC 2023. Lecture Notes in Computer Science, vol. 14446, pp. 295–312. Springer (2023)
3. Baier, C., Katoen, J.P.: Principles of Model Checking. MIT Press (2008)
4. Baldan, P., Crafa, S.: A logic for true concurrency. In: Gastin, P., Laroussinie, F. (eds.) CONCUR 2010 - Concurrency Theory. Lecture Notes in Computer Science, vol. 6269, pp. 147–161. Springer (2010)
5. Baldan, P., Crafa, S.: Hereditary history-preserving bisimilarity: Logics and automata. In: Garrigue, J. (ed.) Programming Languages and Systems. Lecture Notes in Computer Science, vol. 8858, pp. 469–488. Springer (2014)
6. Baldan, P., Padoan, T.: Model checking a logic for true concurrency. ACM Transactions on Computational Logic (TOCL) **21**(4), 1–49 (2020)
7. Bednarczyk, M.A.: Categories of Asynchronous Systems. Ph.D. thesis, University of Sussex (1987)
8. De Nicola, R., Vaandrager, F.: Three logics for branching bisimulation. Journal of the ACM (JACM) **42**(2), 458–487 (1995)
9. Fahrenberg, U., Johansen, C., Struth, G., Ziemiański, K.: Languages of higher-dimensional automata. Mathematical Structures in Computer Science **31**(5), 575–613 (2021). https://doi.org/10.1017/S0960129521000293
10. Fahrenberg, U., Johansen, C., Struth, G., Ziemiański, K.: A Kleene Theorem for Higher-Dimensional Automata. In: Klin, B., Lasota, S., Muscholl, A. (eds.) 33rd International Conference on Concurrency Theory (CONCUR 2022). Leibniz International Proceedings in Informatics (LIPIcs), vol. 243, pp. 29:1–29:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2022). https://doi.org/10.4230/LIPIcs.CONCUR.2022.29
11. Fahrenberg, U., Ziemiański, K.: A Myhill-Nerode theorem for higher-dimensional automata. In: Gomes, L., Lorenz, R. (eds.) Application and Theory of Petri Nets and Concurrency. Lecture Notes in Computer Science, vol. 13929, pp. 167–188. Springer (2023)
12. Fishburn, P.C.: Intransitive indifference with unequal indifference intervals. Journal of Mathematical Psychology **7**(1), 144–149 (1970)
13. van Glabbeek, R.J.: The linear time - branching time spectrum. In: Baeten, J.C.M., Klop, J.W. (eds.) CONCUR '90 Theories of Concurrency: Unification and Extension. Lecture Notes in Computer Science, vol. 458, pp. 278–297. Springer (1990)
14. van Glabbeek, R.J.: On the expressiveness of higher dimensional automata. Theoretical Computer Science **356**(3), 265–290 (2006)
15. van Glabbeek, R.J., Goltz, U.: Refinement of actions and equivalence notions for concurrent systems. Acta Informatica **37**(4), 229–327 (2001)
16. van Glabbeek, R.J., Vaandrager, F.: The difference between splitting in n and n+1. Information and Computation **136**(2), 109–142 (1997)
17. Hennessy, M., Milner, R.: Algebraic laws for nondeterminism and concurrency. Journal of the ACM (JACM) **32**(1), 137–161 (1985)
18. Janicki, R., Koutny, M.: Structure of concurrency. Theoretical Computer Science **112**(1), 5–52 (1993)

19. Joyal, A., Nielsen, M., Winskel, G.: Bisimulation from open maps. Information and Computation **127**(2), 164–185 (1996). https://doi.org/10.1006/inco.1996.0057
20. Leifer, J.J., Milner, R.: Deriving bisimulation congruences for reactive systems. In: Palamidessi, C. (ed.) CONCUR 2000 — Concurrency Theory. Lecture Notes in Computer Science, vol. 1877, pp. 243–258. Springer (2000)
21. Nielsen, M., Clausen, C.: Bisimulation for models in concurrency. In: Jonsson, B., Parrow, J. (eds.) CONCUR '94: Concurrency Theory. Lecture Notes in Computer Science, vol. 836, pp. 385–400. Springer (1994)
22. Nielsen, M., Plotkin, G., Winskel, G.: Petri nets, event structures and domains, part I. Theoretical Computer Science **13**(1), 85–108 (1981)
23. Phillips, I., Ulidowski, I.: Event identifier logic. Mathematical Structures in Computer Science **24**(2) (2014)
24. Pnueli, A., Manna, Z.: The temporal logic of reactive and concurrent systems. Springer **16**, 12 (1992)
25. Pratt, V.: Modeling concurrency with geometry. In: Proceedings of the 18th Symposium on Principles of Programming Languages, POPL'91. p. 311–322. Association for Computing Machinery (1991). https://doi.org/10.1145/99583.99625
26. Sangiorgi, D.: On the bisimulation proof method. Mathematical Structures in Computer Science **8**(5), 447–479 (1998)
27. Shehtman, V.B.: Bisimulation games and locally tabular logics. Russian Mathematical Surveys **71**(5), 979 (2016). https://doi.org/10.1070/RM9731
28. Shields, M.W.: Concurrent machines. The Computer Journal **28**(5), 449–465 (1985)
29. Van Glabbeek, R., Plotkin, G.: Configuration structures. In: 10th Annual IEEE Symposium on Logic in Computer Science, LICS'95. pp. 199–209. IEEE (1995). https://doi.org/10.1109/LICS.1995.523257
30. Winkowski, J.: An algebraic characterization of the behaviour of non-sequential systems. Information Processing Letters **6**(4), 105–109 (1977)
31. Winskel, G.: Event structures. In: Brauer, W., Reisig, W., Rozenberg, G. (eds.) Petri Nets: Applications and Relationships to Other Models of Concurrency. Lecture Notes in Computer Science, vol. 255, pp. 325–392. Springer (1987)
32. Winskel, G.: An introduction to event structures. In: de Bakker, J.W., de Roever, W.P., Rozenberg, G. (eds.) Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency. Lecture Notes in Computer Science, vol. 354, pp. 364–397. Springer (1989)

A Particular case

In this section, we consider S as the empty conclist, meaning that we allow initial cell to be a node.

Lemma 50. *Let* $\alpha = (x_0, \varphi_1, x_1, \varphi_2, \dots, \varphi_n, x_n) \in \mathrm{Path}_{\mathcal{X}}$ *where* $A \subseteq \mathsf{ev}(x_n)$. *There exists* $\alpha' \in \mathrm{Path}_{\mathcal{X}}$ *such that* $\mathsf{end}(\alpha') = \delta^0_A(x_m)$.

Proof. We will show the case where $A = \{a\}$. The general case could be shown by iterating following the same principle and using the relation $\delta^0_{A\cup B} = \delta^0_A\delta^0_B$. Since $|\mathsf{ev}(x_0)| = 0$ and there exists k such that $\varphi_k = \square(U_{k-1}, U_k)$, $\delta_a(x_k) = x_{k-1}$, and $a \in \mathsf{ev}(x_k)$ for all $i \geq k$, so that $\alpha = (x_0, \varphi_1, x_1, \varphi_2, \dots, \varphi_{k-1}, \delta_a(x_k), \varphi_k, x_k, \dots, \varphi_n, x_n)$. Let $\alpha' = \left(x_0, \varphi_1, x_1, \varphi_2, \dots, \varphi_{k-1}, \delta_a(x_k), \delta^0_a(x_k), \dots, \varphi'_n, \delta^0_a(x_n)\right)$ where $\mathsf{ev}(x_k) = U_k$ and for all $i \geq k+1$, either

- $\varphi'_i = d^0_A \in \square\left(U_{i-1} \setminus a, U_i \setminus a\right), A \subseteq U_i \setminus a$ and $\delta^0_a(x_{i-1}) = \delta^0_A\left(\delta^0_a(x_i)\right)$ (up-step), or
- $\varphi'_i = d^1_B \in \square\left(U_i, U_{i-1}\right), B \subseteq U_{i-1}, \delta^1_B\left(\delta^0_a(x_{i-1})\right) = \delta^0_a(x_i)$ (down-step).

We will denote by $\delta^0_A(\alpha)$ the path constructed in the proof.

Theorem 51. *Two HDAs* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are Cell-bisimilar iff they are strong Path-bisimmilar.*

Proof. $\Rightarrow$ This is equivalent to Th. 45 and can be shown in a similar way.

$\Leftarrow$ Assume that there is a Path-bisimulation R between $\mathcal{Y}$ and $\mathcal{Z}$. We can easily check that the relation $\overline{R} = \left\{(\mathsf{end}(\delta^0_A(\alpha)), \mathsf{end}(\delta^0_A(\beta)) \mid (\alpha, \beta) \in R \text{ and } A \subseteq \mathsf{ev}(\mathsf{end}(\alpha))\right\}$ is a Cell-bisimulation

The hierarchy in this case is shown in Fig. 9.

B Omitted Proofs

Proof (Proof of Thm. 19). Let $(f_1, \varepsilon_1), (f_2, \varepsilon_2) : P \to Q$ be morphisms of **IP** such that $\mathsf{Tr}(f_1, \varepsilon_1) = \mathsf{Tr}(f_2, \varepsilon_2) : \square^P \to \square^Q$. Using the expression of the composition

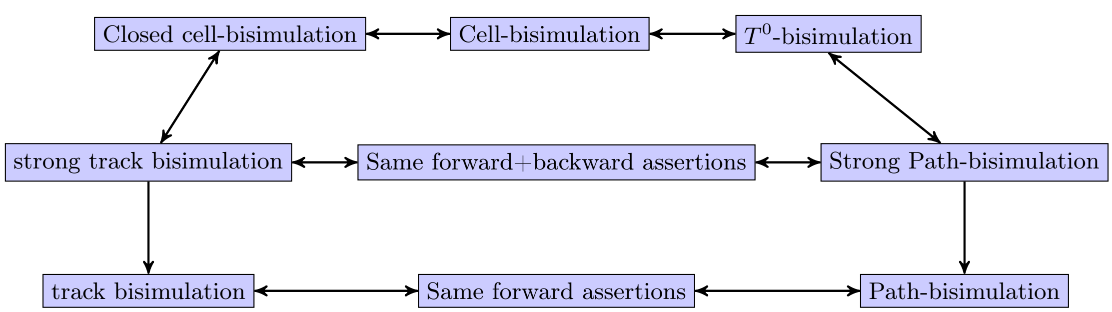


Fig. 9: Hierarchy of notions of equivalence in the particular case $S = \emptyset$.

in Def. 15, we have $(f_1 \circ g, \eta_1) = (f_2 \circ g, \eta_2)$ for all (g, ζ). In particular for $(g, \zeta) = \mathbf{id}^P_{\mathbf{IP}}$, i.e, $g : P \to P$ such that $g(p) = p$ for all p and $\zeta^{-1}(\lrcorner) = P$, we obtain $f_1 = f_2$ and $\eta_1 = \eta_2$, that is

$$\begin{cases} \zeta(f_1^{-1}(q)) & \text{if } q \in f_1(P), \\ \varepsilon_1(q) & \text{otherwise.} \end{cases} = \begin{cases} \zeta(f_2^{-1}(q)) & \text{if } q \in f_2(P), \\ \varepsilon_2(q) & \text{otherwise.} \end{cases}$$

On the other hand, since $\varepsilon_i(q) = \begin{cases} \lrcorner & \text{if } q \in f_i(P), \\ \varepsilon_i(q) & \text{otherwise.} \end{cases}$, $\varepsilon_1 = \varepsilon_2$. Thus $(f_1, \varepsilon_1) = (f_2, \varepsilon_2)$.

Proof (Proof of Prop. 23). We employ induction on m and use Lem. 22.

Proof (Proof of Prop. 33). Induction on the number of cells of α.

- If $\alpha = (x)$, then $\rho'_P = (\mathbf{y}_P)$, and the Yoneda embedding ι_x satisfy the requirements.
- If $\alpha = (y \nearrow^A x)$, then $P = {}_{\mathsf{ev}(x)\setminus A}\mathsf{ev}(x)_{\mathsf{ev}(x)}$ and $\rho'_P = (\delta^0_A(\mathbf{y}_U), \mathbf{y}_U)$, where $U = \mathsf{ev}(x)$. Again, we take $g_\alpha = \iota_x$.
- If $\alpha = (x \searrow^B y)$, then $P = {}_{\mathsf{ev}(x)}\mathsf{ev}(x)_{\mathsf{ev}(y)\setminus B}$ and $\rho'_P = (\mathbf{y}_U, \delta^1_A(\mathbf{y}_U))$, where $U = \mathsf{ev}(x)$. Similarly, we consider $g_\alpha = \iota_x$.
- If $\alpha = \beta * \theta$ where both β and θ are shorter than α. Let $R = \mathsf{ev}(\beta)$ and $Q = \mathsf{ev}(\theta)$ (we therefore have $P = R * Q$). By induction hypothesis, there exist $\rho'_R \simeq \rho_R$, $\rho'_Q \simeq \rho_Q$, a unique $g_\beta : \square^R \to X$, and a unique $g_\theta : \square^Q \to X$, such that $g_\beta(\rho'_R) = \beta$ and $g_\theta(\rho'_Q) = \theta$. $g_\beta * g_\theta$ satisfies the requirement of g_α. By Def. 32, $\rho'_R * \rho'_Q \simeq \rho_P$.

Proof (Proof of Lem. 34). Induction on the length of α.

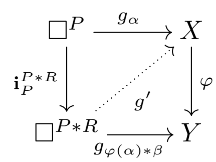
- If $\alpha = (x \nearrow^{A\cup B} z)$ and $\beta = (x \nearrow^A y \nearrow^B z)$, then both g_α and g_β are the Yoneda embedding ι_z.
- Similarly, if $\alpha = (x \searrow_{A\cup B} z)$ and $\beta = (x \searrow_A y \searrow_B z)$.
- If $\theta * \alpha * \gamma \simeq \theta * \beta * \gamma$ such that $\alpha \simeq \beta$. We have $g_{\theta*\alpha*\gamma} = g_\theta * g_\alpha * g_\gamma$ and $g_{\theta*\beta*\gamma} = g_\theta * g_\beta * g_\gamma$. By induction hypothesis, $g_\alpha = g_\beta$, thus $g_{\theta*\alpha*\gamma} = g_{\theta*\beta*\gamma}$. □

Proof (Proof of Thm. 35). Let us define a functor $\Psi : \mathbb{P}_X \to \mathbb{T}_X$ by $\Psi([\alpha]) = g_\alpha$, $\Psi(e^{\alpha*\beta}_\alpha) = \mathbf{e}^{\mathsf{ev}(\alpha)*\mathsf{ev}(\beta)}_{\mathsf{ev}(\alpha)}$. By Lem. 53 in App. B, Υ is a functor. To show that it is the inverse of Ψ, let $\alpha \in \mathbb{P}_X$, $\Upsilon \circ \Psi(\alpha) = \Upsilon(g_\alpha) = g_\alpha(\rho_P) = \alpha$ by Lem. 34 and Prop. 33. We use the same argument to show that $\Upsilon \circ \Psi(e^{\alpha*\beta}_\alpha) = e^{\alpha*\beta}_\alpha$. To show that $\Psi \circ \Upsilon = \mathbf{id}_{\mathbb{T}_X}$, we have $\Psi \circ \Upsilon(p) = \Psi(p(\rho_P)) = g_{p(\rho_P)} = p$ by the uniqueness of such a track (Lem. 34). The same argument applies to show that $\Psi \circ \Upsilon(\mathbf{e}^{P*R}_P) = (\mathbf{e}^{P*R}_P)$.

Lemma 52. *For any HDA map $\varphi : \mathcal{X} \to \mathcal{Y}$ and $\alpha \in \mathrm{Path}_\mathcal{X}$, we have $\varphi \circ g_\alpha = g_{\varphi(\alpha)}$.*

Proof (Proof of Lem. 52). Let $P = \mathsf{ev}(\alpha)$ and ρ'_P the path of Prop. 33. By definition, $g_\alpha(\rho'_P) = \alpha$, thus $\varphi \circ g_\alpha(\rho'_P) = \varphi(\alpha)$. We obtain the equality required by the uniqueness of $g_{\varphi(\alpha)}$.

Proof (Proof of Lem. 37). $1 \Rightarrow 2$ Let $\mathsf{ev}(\alpha) = P$ and $\mathsf{ev}(\beta) = R$. By the definition of $g_{\varphi(\alpha)*\beta}$ (as constructed int the proof of Prop. 33), we have $g_{\varphi(\alpha)*\beta} \circ \mathbf{i}_P^{P*R} = g_{\varphi(\alpha)} = \varphi \circ g_\alpha$ (by Lem. 52), meaning that the following diagram commutes



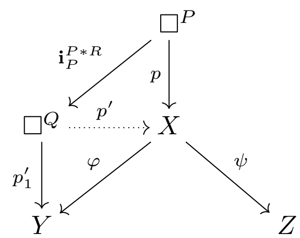
Since φ is $\mathbf{T}^0$-open, there exists $g' : \square^{P*R} \to Y$ such that $g' \circ \mathbf{i}_P^{P*R} = g_\alpha$ and $\varphi \circ g' = g_{\varphi(\alpha)*\beta}$. Consider $\alpha' = g'(\mathbf{f}_Q^{P*Q}(\rho'_R))$. We have $g' \circ \mathbf{i}_P^{P*R}(\rho'_P) = \alpha$, thus α and α' can be concatenated. On the other hand, $\varphi(\alpha') = (g_{\varphi(\alpha)} * g_\beta) \circ \mathbf{f}_Q^{P*Q}(\rho'_R) = g_\beta(\rho'_R) = \beta$, by the definition of the track concatenation.

$2 \Rightarrow 1$ Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be an HDA map, and $q : \square^{P*R} \to Y$ such that $\varphi \circ p = q \circ \mathbf{i}_P^{P*R}$. Let ρ_P and ρ_Q be the characteristic paths of P and Q respectively. By definition, $\rho_Q = \mathbf{i}_P^{P*R}(\rho_P) * \sigma$ for some $\sigma \in \mathrm{Path}_{\square^Q}$, hence $q(\rho_Q) = \big(q \circ \mathbf{i}_P^{P*R}(\rho_P)\big) * q(\sigma)$. Defining $\alpha = p(\rho_P)$ and $\beta = q(\sigma)$, we obtain $q(\rho_Q) = \varphi(\alpha) * \beta$. The *future path lifting property* yields a *path* α' such that $\varphi(\alpha * \alpha') = \varphi(\alpha) * \beta$. Since $\mathsf{ev}(\alpha * \alpha') = \mathsf{ev}(q(\rho_Q)) = Q$, by Prop. 33, there exists a unique track $g : \square^Q \to X$ such that $g(\rho_Q) = \alpha * \alpha'$. On one hand $g \circ \mathbf{i}_P^{P*R}(\rho_P) = \alpha = p(\rho_P)$, thus by Prop. 33, $g \circ \mathbf{i}_P^{P*R} = p$. On the other hand, $\varphi \circ g(\rho_Q) = \varphi(\alpha * \alpha') = q(\rho_Q)$ thus again by Prop. 33 $\varphi \circ g = q$. Therefore, φ is $\mathbf{T}_S^0$-open.

Proof (Proof of Thm. 40). $\Rightarrow$: Assume that there is a span of $\mathbf{T}_S^0$-open HDA-maps $\mathcal{Y} \xleftarrow{\varphi} \mathcal{X} \xrightarrow{\psi} \mathcal{Z}$. The relation $K = \{(\varphi(x), \psi(x)) \mid x \in \mathcal{X}_{acc}\}$ is a Cell-bisimulation. Since $\varphi(i_X) = i_Y$ and $\psi(i_X) = i_Z$, 39. 1 is satisfied. By [9, Lemma 27], K respects labels, thus 39.2 is satisfied. Condition 39.3 is satisfied as φ is a precubical map. To show 39.4 and similarly 39.5, let $(y, z) = (\varphi(x), \psi(x))$. Assume that there exists $y' \in Y$ such that $\delta_A^0(y') = y$, meaning that $\varphi(\alpha_x)$ can be concatenated with $\beta = (y \nearrow^A y')$. Since φ is open, by Lem. 37, there exists $\alpha'_x = (x \nearrow^A x')$ in X such that $\varphi(\alpha') = (y \nearrow^A y')$. Defining $z' = \psi(x')$ we obtain $\delta_A^0(z') = z$ and $(y', z') \in K$.

$\Leftarrow$: Assume that there exists a Cell-bisimulation R between $\mathcal{Y}$ and $\mathcal{Z}$. Let $\mathcal{X} = (X, (i_Y, i_Z))$ where $X = R$ and $\delta_A^\nu(y, z) = (\delta_A^\nu(y), \delta_A^\nu(z))$. Let φ and ψ be projections which for $(y, z) \in X$ give y and z respectively. By condition 39.1, φ and ψ preserve initial cells. Let $(y, z) \in X$, and then an up-step $\beta = \big(\varphi(y, z) \nearrow^A y'\big)$ in Y. As $(y, z) \in \mathsf{R}$, by 39 .4, there exists $z' \in Z$ such that $z \nearrow^A z'$ and $(y', z') \in R$. That is, there exists an up-step $\alpha = ((y, z) \nearrow^A (y', z'))$ in X such that $\varphi(\alpha) = \beta$. Thus, φ has the future path-lifting property. By Lem. 37, φ is $\mathbf{T}_S^0$-open. We can show that ψ is $\mathbf{T}_S^0$-open similarly.

Proof (Proof of Thm. 45). Assume that there is a span of $\mathbf{T}_S^0$-open maps with a common HDA $\mathcal{X}$. We show that the relation $K = \{(\varphi \circ p, \psi \circ p); p \text{ is a track in } X\}$ is a strong Track-bisimulation. Since φ and ψ preserve initial cells, K satisfies 41.1. To show that K satisfies the condition 41.2, assume that $(p_1, p_2) \in K$, so that $(p_1, p_2) = (\varphi \circ p, \psi \circ p)$ for some track p in X. If $p_1 = p'_1 \circ \mathbf{i}_P^{P*R}$ for some morphism $\mathbf{i}_P^{P*R} : \Box^P \to \Box^Q$ of $\mathbf{T}_S^0$, that is, the left square in the following diagram commutes



Then, since φ is $\mathbf{T}_S^0$-open, there exists $p' : \Box^Q \to X$ such that the two triangles in the previous diagram commute. Let $p'_2 = \psi \circ p'$, we have $(p'_1, p'_2) \in K$ and $p'_2 \circ \mathbf{i}_P^{P*R} = \psi \circ p' \circ \mathbf{i}_P^{P*R} = \psi \circ p = p_2$ as required by 41.2. Finally, if $(p_1, p_2) \in K$, it is clear that for any initial inclusion $\mathbf{i}_P^{P*R} : \Box^P \to \Box^Q$ $(p_1 \circ \mathbf{i}_P^{P*R}, p_2 \circ \mathbf{i}_P^{P*R}) \in K$. Thus, the Track-bisimulation K is strong.

Lemma 53. *The map $\Upsilon : \mathbb{T}_X \to \mathbb{P}_X$ given as follows:*
- $\Upsilon(p) = p(\rho_P)$ *for* $p : \Box^P \to X \in \mathbf{Ob}(\mathbb{T}_X)$;
- $\Upsilon(f) = [p(\rho_P)] \to \big[p(\rho_P) * \big(p' \circ \mathbf{f}_R^{P*R}(\rho_R)\big)\big]$ *for* $f : p \to p'$, *where* $p : \Box^P \to X$ *and* $p' : \Box^{P*R} \to X$

is a functor.

Proof. To show that Υ is a functor, on the one hand, we have

$$\Upsilon(\mathbf{e}_P^{P*R*Q})(p) = [p(\rho_P)] \to [p(\rho_P) * p'' \circ \mathbf{f}_{R*Q}^{P*R*Q}(\rho_{R*Q})] \tag{1}$$

$$= [p(\rho_P) * p'' \circ \mathbf{f}_{R*Q}^{P*R*Q}(\mathbf{i}_R^{R*Q}(\rho_R) * \mathbf{f}_Q^{R*Q}(\rho_Q))] \tag{2}$$

$$= \big[p(\rho_P) * \big(p'' \circ \mathbf{f}_{R*Q}^{P*R*Q} \circ \mathbf{i}_R^{R*Q}(\rho_R)\big) * \big(p'' \circ \mathbf{f}_{R*Q}^{P*R*Q} \circ \mathbf{f}_Q^{R*Q}(\rho_Q)\big)\big]. \tag{3}$$

On the other hand, we have[10]

$$\Upsilon(\mathbf{e}_P^{P*R}) = [p(\rho_P)] \to \big[p(\rho_P) * \big(p' \circ \mathbf{f}_R^{P*R}(\rho_R)\big)\big] = [p'(\rho_{P*R})]$$
$$\Upsilon(\mathbf{e}_{P*R}^{L}) = [p'(\rho_{P*R})] \to \big[(p(\rho_P) * p' \circ \mathbf{f}_R^{P*R}(\rho_R)) * p'' \circ \mathbf{f}_Q^{L}(\rho_Q)\big].$$

where $L = P * R * Q$. Thus,

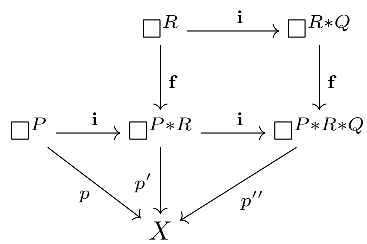
$$\big(\Upsilon(\mathbf{e}_{P*R}^{P*R*Q}) \circ \Upsilon(\mathbf{e}_P^{P*R})\big)(p) = (p(\rho_P) * p' \circ \mathbf{f}_R^{P*R}(\rho_R)) * p'' \circ \mathbf{f}_Q^{P*R*Q}(\rho_Q). \tag{4}$$

To show that $\Upsilon(\mathbf{e}_P^{P*R*Q})(p) = \big(\Upsilon(\mathbf{e}_{P*R}^{P*R*Q}) \circ \Upsilon(\mathbf{e}_P^{P*R})\big)(p)$, we need to show that (1)=(4). To show the equality of the second paths in each equation, we prove that

$$p'' \circ \mathbf{f}_{R*Q}^{P*R*Q} \circ \mathbf{i}_R^{R*Q} = p' \circ \mathbf{f}_R^{P*R} \tag{5}$$

[10] Note that $\Upsilon(f)(p) = p'(\rho_{P*R})$ because $\rho_{P*R} = \mathbf{i}_P^{P*R}(\rho_P) * \mathbf{f}_R^{P*R}(\rho_R)$ and $p' \circ \mathbf{i}_P^{P*R} = p$.

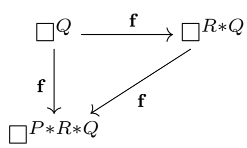
Using the expression of initial and final inclusions (Lem. 22), an elementary calculation shows that $\mathbf{f}_{R*Q}^{P*R*Q} \circ \mathbf{i}_R^{R*Q} = \mathbf{i}_{P*R}^{P*R*Q} \circ \mathbf{f}_R^{P*R}$, meaning that the square in the following diagram commutes. Thus, $p'' \circ \mathbf{f}_{R*Q}^{P*R*Q} \circ \mathbf{i}_R^{R*Q} = p'' \circ \mathbf{i}_{P*R}^{P*R*Q} \circ \mathbf{f}_R^{P*R}$ and (5) follows since $p'' \circ \mathbf{i}_{P*R}^{P*R*Q} = p'$



what left is to show the third paths in each equation are equal, i.e,

$$p'' \circ \mathbf{f}_{R*Q}^{P*R*Q} \circ \mathbf{f}_Q^{R*Q}(\rho_Q) = p'' \circ \mathbf{f}_Q^{P*R*Q}(\rho_Q)$$

which is clear by the following diagram



Proof (Proof of Thm. 49). $\Rightarrow$ Assume that there is a (strong) track-bisimulation K between $\mathcal{X}_1$ and $\mathcal{X}_2$. We show that the relation between tracks with domain $\square^P$ given by $\mathsf{R} = \{(p_1(\rho_P), p_2(\rho_P)) | \ (p_1, p_2) \in K\}$, where ρ_P is the characteristic path of P, is a (strong) path-bisimulation. By Yoneda lemma, there is a one-to-one correspondence between the initial cells and the initial tracks, thus R satisfies 48.1. Since p_i are precubical maps, 48.2 is satisfied. Note that $\mathsf{R} = \{(\Upsilon(p_1), \Upsilon(p_1)) | \ (p_1, p_2) \in K\}$, where Υ is the isomorphism of Lemma 53. The property 48.3 holds because Υ is an isomophism between the categories $\mathbb{P}_{X_i}$ and T_{X_i} (Th. 35) having extensions as morphisms and by the the property 41.2.

$\Leftarrow$ For any path $\alpha \in \mathrm{P}_X$, we denote by $p_\alpha : \square^{\mathsf{ev}(\alpha)} \to X$ the track of Prop. 33. We assume that there is a strong Path-bisimulation R between HDAs $\mathcal{X}_1$ and $\mathcal{X}_2$. We show that the relation between tracks $K = \{(p_{\alpha_1}, p_{\alpha_2}) \mid (\alpha_1, \alpha_2) \in K\}$ is a strong $\mathbf{T}_S^0$-bisimulation. First, K satisfies 41.1 because there is one-to-one correspondence between initial tracks and initial cells (Yoneda embedding). To show 41.2, it is enough to notice that $K = \{(\Psi(\alpha_1), \Psi(\alpha_2)) | \ (\alpha_1, \alpha_2) \in K\}$ and to use the property 48.3.